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CONTENTS

Cyclic Polygons . . . . . W. B. CARVER 533
Circular Probability Problems . . . . . W. C. GUENTHER 541
A General Chain Rule without Components for Derivatives in Vector Spaces . . . . . J. V. LEWIS 545
Necessary and Sufficient Conditions for Prime Pairs R. D. LARSSON 549
The Role of Industrial Members in the Mathematical Association of America . . . . . H. O. POLLAK 551
A Note on a Generalization of Boolean Matrix Theory . . . . . MICHAEL YOELI 552
Mathematical Notes. J. L. BROWN, JR., L. CARLITZ, NEILL MCSHANE 557
Classroom Notes . . . . . ROY LEIPNIK, NORMAN SCHAUMBERGER, BERTHOLD SCHWEIZER, ALLAN DAVIS 563
Mathematical Education Notes 568
Elementary Problems and Solutions 572
Advanced Problems and Solutions 576
Recent Publications 582
News and Notices 586
The Mathematical Association of America 589
The Proposed Doctor of Arts Degree 589
February Meeting of the Louisiana-Mississippi Section . . . 589
March Meeting of the Southern California Section 591
Calendar of Future Meetings 592

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The AMERICAN MATHEMATICAL MONTHLY

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CYCLIC POLYGONS

W. B. CARVER, Cornell University

In the April 1960 issue of the MONTHLY, problem E 1411 was concerned with a cyclic pentagon. The theorem to be proved was a proportion, each term of the proportion being the product of the lengths of five line segments determined by the pentagon. The proposer's solution* gave a proof which he generalized to a similar proportion for a cyclic n-gon, the terms of the proportion being products of n segments. In this paper the problem is attacked by the use of complex coordinates, and a large number of similar relations are obtained for the pentagon and for the n-gon, the number increasing enormously with n. Many of the theorems proved are simpler than the proportion of the problem, the theorems stating equality between two products rather than a proportion between four products.

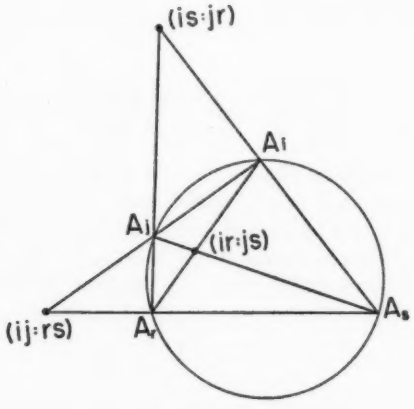

FIG. 1

Consider n distinct points on a circle, $n \geqq 5$, and name them *in any order*, A_i, $i=1, \cdots, n$. We then have a simple n-gon (not in general convex) with vertices at A_i and with n sides, A_1A_2, A_2A_3, $\cdots$, A_nA_1. For a fixed set of n points there are $\frac{1}{2}(n-1)!$ such simple n-gons. The n points, taken in pairs, determine $\frac{1}{2}n(n-1)$ lines A_iA_j, including the n sides. In what follows all subscripts are to be reduced, modulo n, to the set $1, \cdots, n$; and equations *in these subscripts* are to be regarded as congruences, modulo n.

The following notations will be used (see Fig. 1):

---

* This MONTHLY, vol. 67, 1960, p. 1029.

ij will denote the distance (positive) between the points A_i and A_j $(ij=ji)$.

$(ij:rs)$, where i, j, r, s are any four distinct subscripts, will denote the point of intersection of the lines A_iA_j and A_rA_s. (The case in which these lines are parallel will be treated separately. Where $(ij:rs)$ is used it is assumed that the lines are not parallel.) There are $\frac{1}{8}n(n-1)(n-2)(n-3)$ of these intersections, but not necessarily all distinct when $n>5$.

$i(ij:rs)$ will denote the distance (positive) between the points A_i and $(ij:rs)$, and similarly for $j(ij:rs)$, $r(ij:rs)$, and $s(ij:rs)$.

$\overset{n}{\prod}[\ ]$ will denote a product of n factors, the first factor being given in the brackets [ ], and each subsequent factor obtained by increasing each subscript of the preceding factor by one. (The superscript n will be omitted from this symbol when the meaning is clear without it.)

Thus if $n=5$ and the points A, B, C, D, E of problem E 1411 are respectively A_1, A_2, A_3, A_4, A_5, then the points Q, R, S, T, P will be respectively $(13:24)$, $(24:35)$, $(35:41)$, $(41:52)$, $(52:13)$; and the proportion to be proved may be written

$$\frac{\prod[32]}{\prod[14]} = \frac{\prod[2(13:24)]}{\prod[4(13:24)]}. \tag{1}$$

We may take the circle as the unit circle in the complex plane, and then each point A_i will correspond to a complex turn t_i, $|t_i|=1$. If a and c are any two complex numbers, the distance between the corresponding points is $\sqrt{\{(a-c)(\bar{a}-\bar{c})\}}$, and the conjugate of any turn is its reciprocal, $\bar{t}=1/t$. Using these two facts one deduces readily that the distance between A_j and A_r is

$$jr = \pm (t_j - t_r)\sqrt{\left(\frac{-1}{t_jt_r}\right)}.$$

The expression on the right is real, and the sign is chosen to make it positive. (This holds also for the $\pm$ signs in the next six equations.) Then

$$\overset{n}{\prod}[jr] = \pm\sqrt{\{(-1)^n\}}\ \overset{n}{\prod}\left[\frac{t_j - t_r}{t_j}\right].$$

Similarly

$$\overset{n}{\prod}[is] = \pm\sqrt{\{(-1)^n\}}\ \overset{n}{\prod}\left[\frac{t_i - t_s}{t_i}\right],$$

$$\frac{\overset{n}{\prod}[jr]}{\overset{n}{\prod}[is]} = \pm\frac{\overset{n}{\prod}[t_j - t_r]}{\overset{n}{\prod}[t_i - t_s]}. \tag{2}$$

The equations of the lines A_iA_j and A_rA_s in conjugate coordinates are*

$$x + t_it_jy = t_i + t_j \quad \text{and} \quad x + t_rt_sy = t_r + t_s;$$

and solving for x, one finds that the point of intersection of these lines corresponds to the complex number

$$\frac{t_it_j(t_r + t_s) - t_rt_s(t_i + t_j)}{t_it_j - t_rt_s}.$$

(If the lines are parallel, $t_it_j - t_rt_s = 0$, a case to be treated separately.) Then the distance between this point and the point A_r is found to be

$$r(ij:rs) = \pm \frac{(t_r - t_i)(t_r - t_j)}{t_it_j - t_rt_s} \sqrt{\left(\frac{-t_s}{t_r}\right)}.$$

Hence

$$\prod_n [r(ij:rs)] = \pm \sqrt{\{(-1)^n\}} \prod_n \left[\frac{(t_r - t_i)(t_r - t_j)}{t_it_j - t_rt_s}\right].$$

Similarly

$$\prod_n [i(ij:rs)] = \pm \sqrt{\{(-1)^n\}} \prod_n \left[\frac{(t_i - t_r)(t_i - t_s)}{t_it_j - t_rt_s}\right],$$

and

$$\frac{\prod_n [r(ij:rs)]}{\prod_n [i(ij:rs)]} = \pm \frac{\prod_n [(t_r - t_i)(t_r - t_j)]}{\prod_n [(t_i - t_r)(t_i - t_s)]} = \pm \frac{\prod_n [t_j - t_r]}{\prod_n [t_i - t_s]}. \tag{3}$$

From (2) and (3) we have the geometrical relation

$$\frac{\prod_n [jr]}{\prod_n [is]} = \frac{\prod_n [r(ij:rs)]}{\prod_n [i(ij:rs)]}$$

for all values of $n \geqq 5$, and all choices of the four subscripts i, j, r, s from the set $1, \cdots, n$.

In exactly the same way we prove three more similar theorems, and the four theorems may be conveniently given in the form

$$\frac{\prod_n [jr]}{\prod_n [is]} = \frac{\prod_n [r(ij:rs)]}{\prod_n [i(ij:rs)]} = \frac{\prod_n [j(ij:rs)]}{\prod_n [s(ij:rs)]}, \tag{4}$$

* W. B. Carver, The conjugate coordinate system for plane Euclidean geometry, this MONTHLY, vol. 63, No. 9, Part II, 1956.

$$(5)\qquad \frac{\overset{n}{\Pi}[js]}{\overset{n}{\Pi}[ir]} = \frac{\overset{n}{\Pi}[s(ij:rs)]}{\overset{n}{\Pi}[i(ij:rs)]} = \frac{\overset{n}{\Pi}[j(ij:rs)]}{\overset{n}{\Pi}[r(ij:rs)]}.$$

One sees next that, for certain choices of the subscripts i, j, r, s, some of the ratios in (4) and (5) become equal to one, giving equality between the numerators and denominators of some of the fractions. In particular

(I) If $i+s=j+r$, $\overset{n}{\Pi}[js]=\overset{n}{\Pi}[ir]$, and therefore

$$\overset{n}{\Pi}[s(ij:rs)] = \overset{n}{\Pi}[i(ij:rs)], \qquad \overset{n}{\Pi}[j(ij:rs)] = \overset{n}{\Pi}[r(ij:rs)].$$

(I′) If $i+r=j+s$, $\overset{n}{\Pi}[jr]=\overset{n}{\Pi}[is]$, and therefore

$$\overset{n}{\Pi}[r(ij:rs)] = \overset{n}{\Pi}[i(ij:rs)], \qquad \overset{n}{\Pi}[j(ij:rs)] = \overset{n}{\Pi}[s(ij:rs)].$$

(II) If $i+j=r+s$, $\overset{n}{\Pi}[js]=\overset{n}{\Pi}[ir]$, $\overset{n}{\Pi}[jr]=\overset{n}{\Pi}[is]$, and therefore

$$\overset{n}{\Pi}[i(ij:rs)] = \overset{n}{\Pi}[j(ij:rs)] = \overset{n}{\Pi}[r(ij:rs)] = \overset{n}{\Pi}[s(ij:rs)].$$

(III) *If n is even* and if $i+j=2r=2s$, $\overset{n}{\Pi}[ir]=\overset{n}{\Pi}[jr]$, $\overset{n}{\Pi}[is]=\overset{n}{\Pi}[js]$, and therefore

$$\overset{n}{\Pi}[i(ij:rs)] = \overset{n}{\Pi}[j(ij:rs)].$$

Whenever i, j, r, s are chosen to satisfy any of the conditions (I), (I′), (II), or (III), we therefore have simpler theorems, theorems stating equality between two products instead of a proportion between four products.

Consider now the case of the pentagon. For $n=5$, any choice whatever of i, j, r, s will satisfy one of the conditions (I), (I′), or (II). If we take i, j, r, s as 1, 3, 2, 4, we have from (4)

$$(6)\qquad \frac{\Pi[32]}{\Pi[14]} = \frac{\Pi[3(13:24)]}{\Pi[4(13:24)]}.$$

Also, since condition (I) is satisfied, we have

$$(7)\qquad \Pi[4(13:24)] = \Pi[1(13:24)],$$

$$(8)\qquad \Pi[3(13:24)] = \Pi[2(13:24)].$$

From (6) and (8) it follows that

$$\frac{\Pi[32]}{\Pi[14]} = \frac{\Pi[2(13:24)]}{\Pi[4(13:24)]},$$

which proves the theorem of problem E 1411 in the form given by (1). In the

notation of the problem we have proved, not merely the proportion, but the simpler relations,

$$CQ \cdot DR \cdot ES \cdot AT \cdot BP = BQ \cdot CR \cdot DS \cdot ET \cdot AP,$$

$$DQ \cdot ER \cdot AS \cdot BT \cdot CP = AQ \cdot BR \cdot CS \cdot DT \cdot EP.$$

Figures 2 and 3 show these relations. In each of Figures 2–8, the product of the lengths of the dotted line segments is equal to the product of the solid segments.

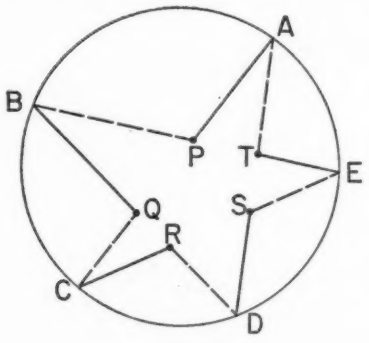

FIG. 2

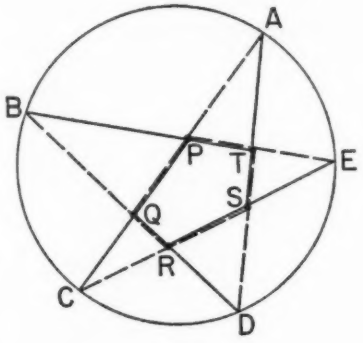

FIG. 3

For $n=5$ there are fifteen points $(ij:rs)$. If we take i, j, r, s as 1, 2, 4, 3, we have a different set of intersections (12:43), (23:54), (34:15), (45:21), and (51:32), condition (I′) is satisfied, and we get a new set of relations similar to (6), (7), and (8), but involving different sets of segments. But a different situation arises when we take i, j, r, s as 1, 4, 3, 2. We have the intersections (14:32), (25:43), (31:54), (42:15), and (53:21), condition (II) is satisfied, and we have equality of four products,

$$\prod [1(14:32)] = \prod [4(14:32)] = \prod [3(14:32)] = \prod [2(14:32)],$$

which is equivalent to three independent relations like (7) and (8). Figures 4 and 5 show the four sets of segments, Figure 4 those of $\prod[1(14:32)]$ and $\prod[3(14:32)]$, and Figure 5, those of $\prod[4(14:32)]$ and $\prod[2(14:32)]$.

We see then that when a simple pentagon has been fixed by assigning the names A_1, A_2, A_3, A_4, A_5 *in some order* to five points on a circle, we have seven different theorems similar to (7) and (8); two such theorems when i, j, r, s are so chosen that condition (I) holds, two different theorems when condition (I′) holds, and three more theorems for condition (II). (Here, and in all that follows, we are not counting the proportional relations like (6) but only the simpler relations like (7) and (8).)

In Figures 2, 3, 4, 5 the names A_i are assigned to the five points to give a *convex* pentagon. But the convex pentagon is only one of twelve different simple pentagons with vertices at these same points, and for each simple pentagon we get seven different theorems. However, we do not get a total of 84 different theorems, but only 42, due to the fact that each theorem is obtained twice. For example, in Figure 6 we use the same points on the circle as in Figure 4, but we assign the names A_i to them in a different order. Then the dotted segments $\prod[3(14:32)]$ in Figure 6 and $\prod[1(14:32)]$ in Figure 4 are the same, and similarly for the solid segments $\prod[4(14:32)]$ in Figure 6 and $\prod[3(14:32)]$ in Figure 4; and hence we do not have two different theorems, but the same theorem occurring twice.

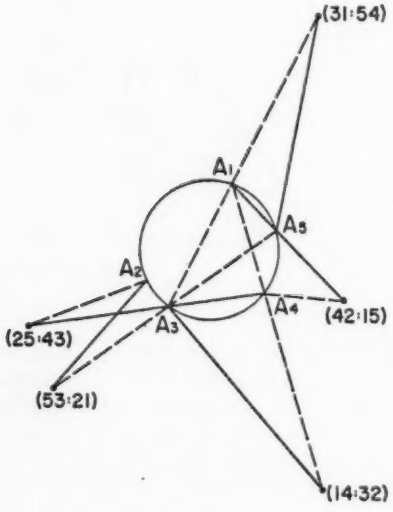

FIG. 4

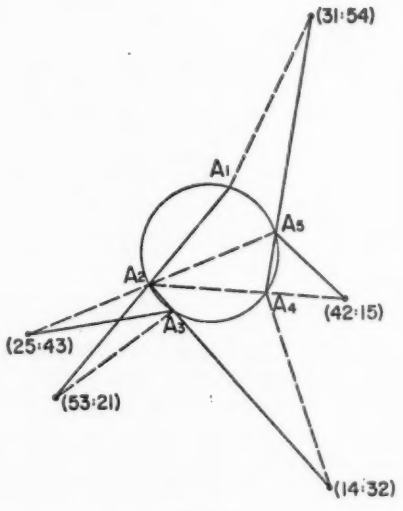

FIG. 5

Since there are $\frac{1}{2}(n-1)!$ different simple n-gons with their vertices at n fixed points on a circle, and since there are $\frac{1}{8}n(n-1)(n-2)(n-3)$ intersections $(ij:rs)$, the number of geometrical relations similar to (7) and (8) increases very rapidly with n. The whole situation is more complicated for composite values of n than for prime values.

For $n=7$ there are 360 simple heptagons with vertices at 7 fixed points, there are 105 intersections $(ij:rs)$, each geometric relation occurs *three* times, and the total number of different geometric relations appears to be 2520. The case $n=6$ presents more difficulties than the case $n=7$, but the total number of different geometric relations is 660. A pretty example is that of the convex hexagon with $(ij:rs)=(13:25)$ so that condition (III) holds. Figure 7 shows this case.

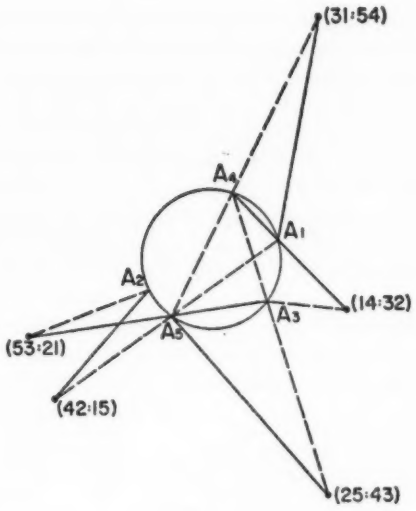

FIG. 6

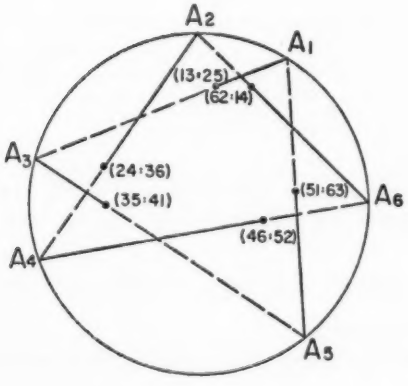

FIG. 7

It remains to consider the case where the point $(ij:rs)$ does not exist because the lines A_iA_j and A_rA_s are parallel. When such a symbol as $j(ij:rs)$ occurs in the numerator of a fraction in (4) or (5) a similar symbol, say $r(ij:rs)$, will occur in the denominator. If one starts with A_iA_j *not parallel* to A_rA_s and moves one of the four points on the circle so that the lines become more and more nearly parallel, the lengths of the segments $j(ij:rs)$ and $r(ij:rs)$ increase indefinitely, but *their ratio obviously approaches one.* Therefore the correct theorem for the parallel

case is obtained by dropping out the symbols $j(ij:rs)$ and $r(ij:rs)$ from the numerator and denominator of the fraction. If several pairs of lines are parallel, we similarly drop out the corresponding factors for each pair of parallels. As an example, take the five points A_i as the points $(-25, 0)$, $(15, -20)$, $(25, 0)$, $(-15, -20)$, and $(-7, 24)$ on the circle $x^2+y^2=625$, with lines A_1A_3 and A_2A_4 parallel and also lines A_4A_1 and A_5A_2. If we take $(ij:rs)$ as $(13:24)$ so that condition (I) holds, (8) may be written

$$\frac{3(13:24)\cdot 4(24:35)\cdot 5(35:41)\cdot 1(41:52)\cdot 2(52:13)}{2(13:24)\cdot 3(24:35)\cdot 4(35:41)\cdot 5(41:52)\cdot 1(52:13)} = 1.$$

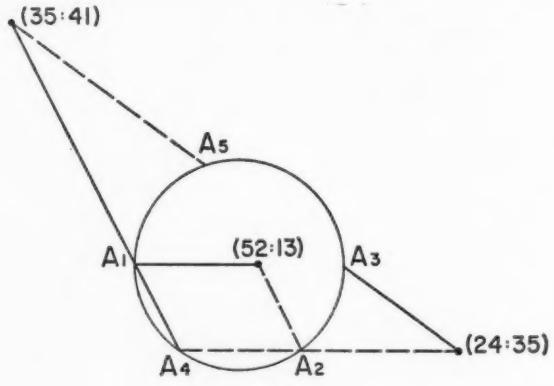

FIG. 8

If we simply drop the pairs of meaningless symbols $3(13:24)$, $2(13:24)$ and $1(41:52)$, $5(41:52)$ from the numerator and denominator, we have the simple relation

$$4(24:35)\cdot 5(35:41)\cdot 2(52:13) = 3(24:35)\cdot 4(35:41)\cdot 1(52:13).$$

Figure 8 shows the segments.

This last example might be given to a high school geometry class in the form:

Let $ABCD$ be any parallelogram with a circle through A, B, and C. Let the lines CD and AD cut the circle at E and F respectively, and let the line EF cut the line BA at G and the line BC at H. Prove that $AD\cdot FH\cdot GB = DC\cdot BH\cdot GE$.

CIRCULAR PROBABILITY PROBLEMS

WILLIAM C. GUENTHER, The Martin Company and University of Wyoming

1. Introduction. If a circle C_1 of radius R is dropped upon a fixed circle C_2 of radius D, several interesting and useful probability problems arise. Two of these involve (a) the probability that C_1 covers a randomly selected point within C_2, and (b) the probability that C_1 covers a randomly selected point on the circumference of C_2. The latter of these two, which will be referred to as the arc-length problem, will be considered in detail. The first, which might be called the area problem, is discussed in [3].

It is assumed that the center of C_1 is aimed at the center of C_2 with aiming errors being circularly normally distributed with unit standard deviation. If the center of C_2 is chosen as the origin, then the center of C_1, say (x, y), has as its probability density function

$$f(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}.$$

The density of $r=\sqrt{(x^2+y^2)}$ is, of course, $g(r)=re^{-r^2/2}$, $r>0$.

2. The arc-length problem. For a given value of r, let $L(r)$ be the length of the arc of C_2 covered by C_1. Then for this value of r the probability sought is $L(r)/(2\pi D)$. Hence the desired probability is

$$P(R, D) = \int_0^\infty g(r) \frac{L(r)}{2\pi D} dr.$$

By using a little elementary geometry, one easily finds that

$$\text{(a)}\quad D > R\text{: } L(r) = \begin{cases} 2D \arccos\left(\dfrac{D^2 + r^2 - R^2}{2rD}\right), & D - R < r < D + R; \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{(b)}\quad D = R\text{: } L(r) = \begin{cases} 2D \arccos\left(\dfrac{r}{2D}\right), & 0 < r < 2D; \\ 2D\pi, & r = 0; \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{(c)}\quad D < R\text{: } L(r) = \begin{cases} 2D \arccos\left(\dfrac{D^2 + r^2 - R^2}{2rD}\right), & R - D < r < R + D; \\ 2D\pi, & 0 \leqq r \leqq R - D; \\ 0, & \text{otherwise.} \end{cases}$$

Hence it follows that

$$P(R, D) = \frac{1}{\pi}\int_{D-R}^{D+R} re^{-r^2/2} \arccos\left(\frac{D^2 + r^2 - R^2}{2rD}\right) dr, \qquad D > R;$$

$$P(R, D) = \frac{1}{\pi}\int_{0}^{2D} re^{-r^2/2} \arccos\left(\frac{r}{2D}\right) dr, \qquad D = R;$$

$$P(R, D) = 1 - e^{-(R-D)^2/2} + \frac{1}{\pi}\int_{R-D}^{R+D} re^{-r^2/2} \arccos\left(\frac{D^2 + r^2 - R^2}{2rD}\right) dr, \quad D < R.$$

3. Evaluation of integrals. First consider the case $D=R$ since it is the simplest of the three. Integration by parts yields

$$P(R, D) = \frac{1}{\pi}\left[\frac{\pi}{2} - \int_0^{2D} \frac{e^{-r^2/2}}{\sqrt{(4D^2 - r^2)}}\, dr\right].$$

Now let $r = 2D \sin \frac{1}{2}\theta$. The result is

$$P(R, D) = \frac{1}{2}\left[1 - \frac{1}{\pi}\int_0^{\pi} e^{-2D^2 \sin^2 \frac{1}{2}\theta} d\theta\right]$$

(1)

$$= \frac{1}{2}\left[1 - \frac{1}{\pi} e^{-D^2}\int_0^{\pi} e^{D^2 \cos\theta} d\theta\right] = \frac{1}{2}\left[1 - e^{-D^2} I_0(D^2)\right], \quad D > 0,$$

where $I_0(x)$ is the modified Bessel function of the first kind of order zero. Tables of $e^{-x}I_0(x)$ are available. Watson [4] has a table covering x from 0 to 16 in increments of .02. For larger x,

$$e^{-x}I_0(x) \cong \frac{1}{\sqrt{(2\pi x)}}\left[1 + \frac{1^2}{1!8x} + \frac{1^2 \cdot 3^2}{2!(8x)^2} + \cdots\right],$$

which is a rapidly converging series [1].

When $D \neq R$ the evaluation is more difficult. Next consider the case $D > R$. Integration by parts yields

$$P(R, D) = \frac{1}{\pi}\int_{D-R}^{D+R} \frac{e^{-r^2/2}(D^2 - R^2 - r^2)}{r\sqrt{[(2rD)^2 - (D^2 + r^2 - R^2)^2]}}\, dr.$$

Letting $r^2 = u^2(D^2 - R^2)$, the integral becomes

$$\frac{1}{\pi}\int_{k^{-1}}^{k} \frac{e^{-cu^2}(1 - u^2)du}{u\sqrt{[(k^2 - u^2)(u^2 - k^{-2})]}} = \frac{1}{2\pi}\int_{k^{-2}}^{k^2} \frac{e^{-ct}(1 - t)dt}{t\sqrt{[(k^2 - t)(t - k^{-2})]}},$$

where $k^2 = (D+R)/(D-R)$, $c = \frac{1}{2}(D^2 - R^2)$. This in turn may be written as two integrals

$$\frac{1}{2\pi}\int_a^b \frac{e^{-ct}dt}{t\sqrt{[(b - t)(t - a)]}} - \frac{1}{2\pi}\int_a^b \frac{e^{-ct}dt}{\sqrt{[(b - t)(t - a)]}} = F - G,$$

where b and a have replaced k^2 and k^{-2} to facilitate writing. In G the substitution $t=\frac{1}{2}(b+a)-\frac{1}{2}(b-a)\cos\theta$ yields

$$G = e^{-\frac{1}{2}c(a+b)}\cdot\frac{1}{2\pi}\int_0^{\pi} e^{\frac{1}{2}c(b-a)\cos\theta}d\theta = \tfrac{1}{2}e^{-\frac{1}{2}c(a+b)}I_0[\tfrac{1}{2}c(b-a)].$$

Next note that $dF/dc=-G$ and $F=\int_c^{\infty}Gdc$. Thus

$$F = \int_c^{\infty}\{\tfrac{1}{2}e^{-\frac{1}{2}(a+b)u}I_0[\tfrac{1}{2}(b-a)u]\}du.$$

It is well known that

$$I_0(x) = \sum_{i=0}^{\infty}\frac{(\frac{1}{2}x)^{2i}}{(i!)^2}.$$

Making this replacement in F yields

$$F = \frac{1}{2}\int_0^{\infty}e^{-\frac{1}{2}(a+b)u}I_0[\tfrac{1}{2}(b-a)u]du - \frac{1}{2}\int_0^{c}e^{-\frac{1}{2}(a+b)u}I_0[\tfrac{1}{2}(b-a)u]du$$

$$= \frac{1}{2\sqrt{(ab)}} - \frac{1}{2}\int_0^{c}\left\{e^{-\frac{1}{2}(a+b)u}\sum_{i=0}^{\infty}\frac{[\frac{1}{4}(b-a)]^{2i}u^{2i}}{(i!)^2}\right\}du.$$

The value of the first integral may be found in any treatise on Bessel functions. In the second integral let $v=\frac{1}{2}(a+b)u$. Then

$$F = \frac{1}{2\sqrt{(ab)}} - \frac{1}{a+b}\sum_{i=0}^{\infty}\left[\frac{b-a}{2(a+b)}\right]^{2i}\frac{(2i)!}{(i!)^2}\int_0^{\frac{1}{2}c(a+b)}\frac{v^{2i}e^{-v}}{(2i)!}dv$$

$$= \frac{1}{2} - \frac{D^2-R^2}{2(R^2+D^2)}\sum_{i=0}^{\infty}\left[\frac{RD}{R^2+D^2}\right]^{2i}\binom{2i}{i}I\left(\frac{R^2+D^2}{2\sqrt{(2i+1)}},\ 2i\right),$$

where $I(u,\ p)$ is the incomplete gamma function which is tabulated in [2]. Finally

$$P(R,\ D) = \frac{1}{2}\left\{1 - e^{-(R^2+D^2)/2}I_0(RD)\right.$$

$$\left. - \frac{D^2-R^2}{R^2+D^2}\sum_{i=0}^{\infty}\left[\frac{2RD}{R^2+D^2}\right]^{2i}\left[\frac{1}{2^{2i}}\binom{2i}{i}\right]I\left(\frac{R^2+D^2}{2\sqrt{(2i+1)}},\ 2i\right)\right\}. \tag{2}$$

Only a few changes are necessary for the case $D<R$. Integration by parts yields

$$P(R,\ D) = 1 - \frac{1}{\pi}\int_{R-D}^{R+D}\frac{e^{-r^2/2}(r^2+R^2-D^2)}{r\sqrt{[(2rD)^2-(D^2+r^2-R^2)^2]}}dr.$$

Letting $r^2=u^2(R^2-D^2)$ the result is

$$P(R, D) = 1 - \frac{1}{\pi}\int_{k^{-1}}^{k} \frac{e^{-cu}(1+u^2)}{u\sqrt{[(k^2-u^2)(u^2-k^{-2})]}}\,du$$

$$= 1 - \frac{1}{2\pi}\int_{k^{-1}}^{k^2} \frac{e^{-ct}(1+t)}{t\sqrt{[(k^2-t)(t-k^{-2})]}}\,dt,$$

where $c=\frac{1}{2}(R^2-D^2)$, $k^2=(R+D)/(R-D)$. Thus $P(R, D)=1-[F+G]$, where the roles of R and D have been interchanged. This reduces to (2). The term containing the infinite series is now added since $R>D$.

Although each case was handled separately, (2) holds for all three situations.

From a practical point of view $R=0$ does not make sense. On the other hand, $D=0$ is an important special case but for this situation $L(r)$ is meaningless. We note that $P(R, 0)=\Pr[r<R]=1-e^{-R^2/2}$.

4. A sample table. By using (2) the entries for Table 1 were prepared. Linear interpolation was used to determine the third decimal place for the incomplete gamma values. Convergence of the series is quite rapid. For values of R and D used for the table no more than nine terms were required to eliminate the contribution to the third decimal place and often considerably less was sufficient.

TABLE 1

$P(R, D)$

D \ R	.5	1.0	1.5	2.0	2.5	3.0	3.5
0	.12	.39	.68	.86	.96	.99	1.00
.5	.10	.36	.63	.83	.94	.98	1.00
1.0	.07	.27	.51	.73	.88	.96	.99
1.5	.04	.16	.36	.58	.77	.90	.96
2.0	.02	.08	.21	.40	.61	.79	.90
2.5	.01	.03	.10	.23	.42	.62	.80
3.0	.00	.01	.04	.11	.25	.43	.63
3.5	.00	.00	.01	.05	.12	.26	.44

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A GENERAL CHAIN RULE WITHOUT COMPONENTS FOR DERIVATIVES IN VECTOR SPACES*

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1. Introduction. Ordinarily the chain rule for derivatives of composite functions in vector spaces must be stated in terms of partial derivatives and components. For example let $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ be functions with the domain of $\mathbf{f}$ in m-dimensional vector space E_m, the range of $\mathbf{f}$ and domain of $\mathbf{g}$ in n-dimensional vector space E_n, and the range of $\mathbf{g}$ in p-dimensional vector space E_p. Let $\mathbf{h}(\mathbf{x})=\mathbf{g}(\mathbf{f}(\mathbf{x}))$ for every $\mathbf{x}\in E_m$. The regular chain rule is used only for real-valued functions g and h (case $p=1$) or for each component of $\mathbf{h}$ separately. It is usually written as

$$\frac{\partial h}{\partial x_r} = \sum_{k=1}^{n} \frac{\partial g}{\partial y_k}\frac{\partial f_k}{\partial x_r}, \qquad r = 1, \cdots, m;$$

or as the scalar vector product

$$\frac{\partial h}{\partial x_r} = \nabla g \cdot \frac{\partial \mathbf{f}}{\partial x_r}, \qquad r = 1, \cdots, m,$$

where $\mathbf{f}$ and h are evaluated at $\mathbf{x}$ and g at $\mathbf{y}$ where $\mathbf{y}=\mathbf{f}(\mathbf{x})$. The chain rule developed here gives the derivative of the vector-valued composite function $\mathbf{h}$ in any direction as a product of a derivative of $\mathbf{g}$ and a derivative of $\mathbf{f}$ and is capable of a simple intuitive interpretation.

2. General chain rule. Use the following notation:

$D_{\mathbf{u}}\mathbf{f}(\mathbf{x})$ = derivative of $\mathbf{f}$ in direction $\mathbf{u}$ at $\mathbf{x}$,
$\mathbf{v}$ = unit vector in direction $D_{\mathbf{u}}\mathbf{f}(\mathbf{x})$,
$D_{\mathbf{v}}\mathbf{g}(\mathbf{y})$ = derivative of $\mathbf{g}$ in direction $\mathbf{v}$ at $\mathbf{y}$,
$\mathbf{h}(\mathbf{x})=\mathbf{g}(\mathbf{f}(\mathbf{x}))$ for every $\mathbf{x}$ in domain $\mathbf{f}$,
$D_{\mathbf{u}}\mathbf{h}(\mathbf{x})$ = derivative of $\mathbf{h}$ in direction $\mathbf{u}$ at $\mathbf{x}$.

Under the conditions given in the theorem in Section 5,

$$D_{\mathbf{u}}\mathbf{h}(\mathbf{x}) = D_{\mathbf{v}}\mathbf{g}(\mathbf{y}) \, \big| \, D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) \, \big| ,$$

where $\mathbf{y}=\mathbf{f}(\mathbf{x})$. Thus the derivative of the composite $\mathbf{h}$ in any direction is the magnitude of the rate of change of $\mathbf{f}$ in that direction times the rate of change of $\mathbf{g}$ in the direction in which $\mathbf{f}$ is changing.

3. Illustrations. *Case* $m=n=p=2$. Let $\mathbf{x}=(x_1, x_2)$, $\mathbf{y}=(y_1, y_2)$ and

$$\mathbf{f}(\mathbf{x}) = (-x_1, 2(x_2+1)x_1) \qquad \text{for every } \mathbf{x} \in E_2,$$
$$\mathbf{g}(\mathbf{y}) = (-y_2y_1, y_1) \qquad \text{for every } \mathbf{y} \in E_2,$$

* Presented to the meeting of the Southwestern Section of the Association, April, 1960.

$$\mathbf{h}(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x})) \qquad \text{for every } \mathbf{x} \in E_2,$$

$$\mathbf{h}(\mathbf{x}) = (2(x_2 + 1)x_1^2, -x_1) \qquad \text{for every } \mathbf{x} \in E_2.$$

Choose $\mathbf{x} = (2, 0)$, $\mathbf{u} = (2/\sqrt{5}, 1/\sqrt{5})$. The derivative of $\mathbf{h}$ in direction $\mathbf{u}$ at $\mathbf{x}$ will be found. This is the ordinary derivative with respect to s at $s = 0$ (along the line consisting of points of the form $\mathbf{x} + s\mathbf{u} = (2 + (2s)/\sqrt{5}, s/\sqrt{5})$ for real s. $\mathbf{f}$ transforms this line into a parabola consisting of points of the form (Fig. 1)

$$\mathbf{f}(\mathbf{x} + s\mathbf{u}) = (-2[1 + s/\sqrt{5}], 4[1 + s/\sqrt{5}]^2)$$

for real s. Finally, $\mathbf{g}$ transforms the parabola into a cubic consisting of points of the form (Fig. 2)

$$\mathbf{g}(\mathbf{f}(\mathbf{x} + s\mathbf{u})) = (8[1 + s/\sqrt{5}]^3, -2[1 + s/\sqrt{5}])$$

for any real s.

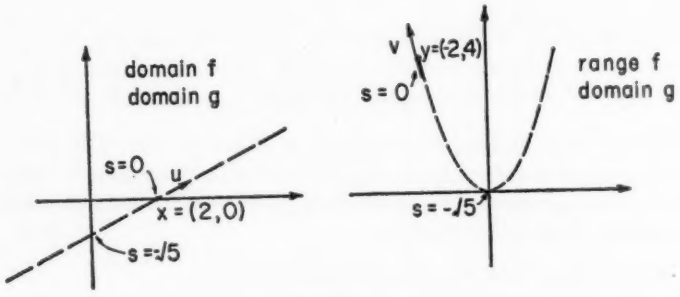

FIG. 1

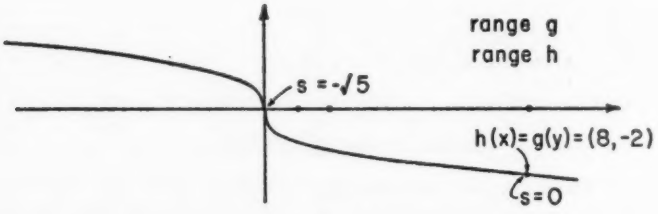

FIG. 2

$$D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) = (-2/\sqrt{5}, 8/\sqrt{5}), \qquad |D_{\mathbf{u}}\mathbf{f}(\mathbf{x})| = 2\sqrt{(17/5)};$$

$$\mathbf{v} = (-1/\sqrt{17}, 4/\sqrt{17}), \qquad D_{\mathbf{v}}\mathbf{g}(\mathbf{y}) = (12/\sqrt{17}, -1/\sqrt{17}), \text{ where } \mathbf{y} = \mathbf{f}(\mathbf{x});$$

$$D_{\mathbf{u}}\mathbf{h}(\mathbf{x}) = D_{\mathbf{v}}\mathbf{g}(\mathbf{y})\,|\,D_{\mathbf{u}}\mathbf{f}(\mathbf{x})| = (12/\sqrt{17}, -1/\sqrt{17}) \cdot 2\sqrt{(17/5)} = (24/\sqrt{5}, -2/\sqrt{5}),$$

which is the direction tangent to the cubic at $\mathbf{h}(\mathbf{x}) = (8, -2)$.

Case $m=n=p=1$. The general chain rule takes an unusual form in one dimension because there are only two directions $+1$ and -1 and we customarily take derivatives only in the $+1$ direction. Let

$$
\begin{aligned}
f(x) &= e^{-x} && \text{for every } x \in E_1,\\
g(y) &= 2y - 1 && \text{for every } y \in E_1,\\
h(x) &= g(f(x)) && \text{for every } x \in E_1,\\
h(x) &= 2e^{-x} - 1 && \text{for every } x \in E_1.\\
D_{+1}f(0) &= -1, \qquad f'(0) = -1;\\
D_{-1}g(1) &= -2, \qquad g'(1) = 2;\\
D_{+1}h(0) &= D_{-1}g(1)\,|\,D_{+1}f(0)\,|, && h'(0) = g'(1)f'(0);\\
D_{+1}h(0) &= (-2)(+1), && h'(0) = (+2)(-1);\\
D_{+1}h(0) &= -2, && h'(0) = -2.
\end{aligned}
$$

The ordinary chain rule uses derivatives in only the positive direction whereas the general chain rule obtains the correct result by taking the derivative of g in the negative direction since f is decreasing.

4. Properties of directional derivatives and the differential. To facilitate proving the general chain rule we state some of the usual definitions and properties of the directional derivative and the differential.

DEFINITION. *The derivative of* $\mathbf{f}$ *in direction* $\mathbf{u}$ (*a unit vector*) *at* $\mathbf{x}$ *is*

$$D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) = \lim_{s\to 0} \frac{\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x})}{s}$$

whenever this limit exists.

DEFINITION. *A function* $\mathbf{f}$ *has a differential* $d\mathbf{f}_{\mathbf{x}}$ *at* $\mathbf{x}$ *provided* $d\mathbf{f}_{\mathbf{x}}$ *is a linear function and for every* $\epsilon>0$ *there is radius* $\delta>0$ *so that for every* $\mathbf{y}$ *for which* $0<|\mathbf{y}-\mathbf{x}|<\delta$, *the following inequality holds*

$$|\,\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) - d\mathbf{f}_{\mathbf{x}}(\mathbf{y}-\mathbf{x})\,| < \epsilon\,|\,\mathbf{y}-\mathbf{x}\,|.$$

Properties. Let $\mathbf{f}$ *have a differential* $d\mathbf{f}_{\mathbf{x}}$ *at* $\mathbf{x}$. *Then*

(1) *If* $\mathbf{u}$ *is a unit vector in domain of* $\mathbf{f}$ *and* $\mathbf{t}=|\mathbf{t}|\mathbf{u}$, $d\mathbf{f}_{\mathbf{x}}(\mathbf{u})=D_{\mathbf{u}}\mathbf{f}(\mathbf{x})$, $d\mathbf{f}_{\mathbf{x}}(\mathbf{t})=D_{\mathbf{u}}\mathbf{f}(\mathbf{x})\,|\mathbf{t}|$.

(2) *For all vectors* $\mathbf{s}$ *and* $\mathbf{t}$ *and all real numbers* a *and* b $d\mathbf{f}_{\mathbf{x}}(a\mathbf{s}+b\mathbf{t})=ad\mathbf{f}_{\mathbf{x}}(\mathbf{s})+bd\mathbf{f}_{\mathbf{x}}(\mathbf{t})$.

(3) *For each* $\mathbf{x}$ *there exists a constant* M *so that* $|D_{\mathbf{u}}\mathbf{f}(\mathbf{x})|<M$ *for every unit vector* $\mathbf{u}$ *in the domain of* $\mathbf{f}$.

(4) *There exists a constant* N *and a radius* $r>0$ *so that for every* $\mathbf{y}$ *in the domain of* $\mathbf{f}$ *if* $0<|\mathbf{y}-\mathbf{x}|<\mathrm{r}$, *then* $|\,\mathbf{f}(\mathbf{y})-\mathbf{f}(\mathbf{x})\,|<N\,|\,\mathbf{y}-\mathbf{x}\,|$.

5. THEOREM. *Let* $\mathbf{f}, \mathbf{g}, \mathbf{h}$ *be functions with domain* $\mathbf{f} \subset E_m$, *range* $\mathbf{f} \subset$ *domain* $\mathbf{g} \subset E_n$, *range* $\mathbf{g} \subset E_p$. *Let* $\mathbf{h}(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$ *for every* $\mathbf{x} \in$ *domain* $\mathbf{f}$. *Let* $\mathbf{f}$ *have a differential at a particular point* $\mathbf{x}$ *and* $\mathbf{g}$ *have a differential at* $\mathbf{y} = \mathbf{f}(\mathbf{x})$. *Let* $\mathbf{u}$ *be a unit vector in* E_m *and* $\mathbf{v}$ *be a unit vector in* E_n *in the direction of* $D_{\mathbf{u}}\mathbf{f}(\mathbf{x})$. *Then*

$$D_{\mathbf{u}}\mathbf{h}(\mathbf{x}) = D_{\mathbf{v}}\mathbf{g}(\mathbf{y}) \left| D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) \right| .$$

Proof. As a preliminary, use the properties to choose a constant $M > 0$ and for each given $\epsilon_1 > 0$ choose $r > 0$ so that for every s, every $\mathbf{z}$ and every $\mathbf{w}$:

(a) If $|s| \leqq r$, then $|\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x}) - d\mathbf{f}_{\mathbf{x}}(s\mathbf{u})| \leqq \epsilon_1 |s|$.
(b) If $|s| \leqq r$, then $|\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x})| \leqq M|s|$.
(c) If $|\mathbf{z}-\mathbf{y}| \leqq r$, then $|\mathbf{g}(\mathbf{z}) - \mathbf{g}(\mathbf{y}) - d\mathbf{g}_{\mathbf{y}}(\mathbf{z}-\mathbf{y})| \leqq \epsilon_1 |\mathbf{z}-\mathbf{y}|$.
(d) If $\mathbf{w}$ is a unit vector in E_n, $|D_{\mathbf{w}}\mathbf{g}(\mathbf{y})| \leqq M$.

It must be shown that

$$\frac{\mathbf{g}(\mathbf{f}(\mathbf{x}+s\mathbf{u})) - \mathbf{g}(\mathbf{f}(\mathbf{x}))}{s} - D_{\mathbf{v}}\mathbf{g}(\mathbf{y}) \left| D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) \right|$$

tends to $\mathbf{0}$ as s tends to 0. Consider the product of this expression with s and later divide by s. Also note that

$$sD_{\mathbf{v}}\mathbf{g}(\mathbf{y}) \left| D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) \right| = d\mathbf{g}_{\mathbf{y}}(sD_{\mathbf{u}}\mathbf{f}(\mathbf{x})) = d\mathbf{g}_{\mathbf{y}}(d\mathbf{f}_{\mathbf{x}}(s\mathbf{u})).$$

Now write

$$\begin{aligned}
&\mathbf{g}(\mathbf{f}(\mathbf{x}+s\mathbf{u})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) - sD_{\mathbf{v}}\mathbf{g}(\mathbf{y}) \left| D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) \right| \\
&\quad = \mathbf{g}(\mathbf{f}(\mathbf{x}+s\mathbf{u})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) - d\mathbf{g}_{\mathbf{y}}(d\mathbf{f}_{\mathbf{x}}(s\mathbf{u})) \\
&\quad = \mathbf{g}(\mathbf{f}(\mathbf{x}+s\mathbf{u})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) - d\mathbf{g}_{\mathbf{y}}(\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x})) \\
&\qquad + d\mathbf{g}_{\mathbf{y}}(\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x})) - d\mathbf{g}_{\mathbf{y}}(d\mathbf{f}_{\mathbf{x}}(s\mathbf{u})) = \mathbf{T}_1 + \mathbf{T}_2,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{T}_1 &= \mathbf{g}(\mathbf{f}(\mathbf{x}+s\mathbf{u})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) - d\mathbf{g}_{\mathbf{y}}(\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x})), \\
\mathbf{T}_2 &= d\mathbf{g}_{\mathbf{y}}(\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x}) - d\mathbf{f}_{\mathbf{x}}(s\mathbf{u})).
\end{aligned}$$

Then $|\mathbf{T}_1| \leqq \epsilon_1 |\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x})|$ provided $|\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x})| \leqq r$ by (c). By (b), $|\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x})| \leqq M|s|$ provided $|s| \leqq r$. Choose δ as the smaller of r and r/M. Then for $|s| < \delta$, $|\mathbf{T}_1| \leqq \epsilon_1 M |s|$. Also $\mathbf{T}_2 = D_{\mathbf{w}}\mathbf{g}(\mathbf{y}) |\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x}) - d\mathbf{f}_{\mathbf{x}}(s\mathbf{u})|$, where $\mathbf{w}$ is a unit vector in the direction of $\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x}) - d\mathbf{f}_{\mathbf{x}}(s\mathbf{u})$, and $|\mathbf{T}_2| \leqq M |\mathbf{f}(\mathbf{x}+s\mathbf{u}) - \mathbf{f}(\mathbf{x}) - d\mathbf{f}_{\mathbf{x}}(s\mathbf{u})|$ by (d). By (a) for $|s| < r$, $|\mathbf{T}_2| \leqq M\epsilon_1 |s|$. Thus,

$$\text{if } |s| \leqq \delta, \quad \text{then} \quad |\mathbf{T}_1 + \mathbf{T}_2| \leqq 2M\epsilon_1 |s| .$$

Given $\epsilon > 0$ choose $\epsilon_1 < \frac{1}{2}(\epsilon/M)$. If $0 < |s| < \delta$, then

$$|\mathbf{g}(\mathbf{f}(\mathbf{x}+s\mathbf{u})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) - sD_{\mathbf{v}}\mathbf{g}(\mathbf{y}) \left| D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) \right| | = |\mathbf{T}_1 + \mathbf{T}_2| \leqq 2M\epsilon_1 |s| < \epsilon |s|.$$

If $0 < |s| < \delta$, then, dividing by $|s|$,

$$\left| \frac{g(f(x + su)) - g(f(x))}{s} - D_v g(y) \, | \, D_u f(x) \, | \right| < \epsilon.$$

Thus the limit of the expression on the left hand side of the above inequality as s tends to 0 is 0. Consequently, $D_u h(x) = D_v g(y) \, | \, D_u f(x) \, |$.

NECESSARY AND SUFFICIENT CONDITIONS FOR PRIME PAIRS

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The author has found a new formulation for the sieve of Eratosthenes, in an attempt to gain fresh insights into some of the problems of prime numbers. In particular, it has been of use in finding new information regarding the existence of an infinite number of prime pairs. The new viewpoint for the sieve will be given first, followed by the theorems concerning prime pairs.

I. PROBLEM. *Given the set of primes* $p_1, p_2, \cdots, p_n, \cdots$, *where* $p_1 = 2$, $p_2 = 3, \cdots$, *find the primes which lie between* p_n *and* p_n^2.

Procedure. Compute r_i for

$$p_n \equiv r_i \pmod{p_i}, \qquad i = 2, \cdots, n-1,$$

where r_i is the least positive residue. We see that $p_n - r_i + kp_i$ is divisible by p_i for $i = 2, \cdots, n-1$ and $k = 1, 2, \cdots$. Now form the set of even integers in the set $kp_i - r_i$. Call this set E. Let E' be the set of positive even integers not contained in E. Set E' has a least element, say b_1. Then $p_n + b_1$ is the next prime p_{n+1}.

Proof. $p_n + b_1$ is not divisible by p_i for $i = 1, \cdots, n-1$ by construction. $p_n + b_1$ is not divisible by p_n unless E' is empty, or $b_1 = p_n^2 - p_n$. But either of these possibilities would require that there be no primes between p_n and p_n^2. This would contradict Bertrand's conjecture [1] that for any positive integer n there is a prime p such that $n < p \leqq 2n$. And certainly $2n < n^2$ for $n > 2$. Now $p_n + b_i$ is a prime for all

$$b_i < p_n^2 - p_n, \qquad b_i \text{ in } E'.$$

The set $p_n + b_i$, $i = 1, 2, \cdots$, represents all primes between p_n and p_n^2, where b_i takes on all values in the set E' which are less than $p_n^2 - p_n$. This set can be ordered of course.

* National Science Foundation Faculty Fellow.

Note the suitability of this method for a program for a high-speed computer, especially for large primes.

II. From I we have

$$p_n \equiv r_i \pmod{p_i} \quad \text{or} \quad p_n - r_i = k_i p_i,$$

where $k_i = [p_n/p_i] \geqq 1$. Therefore $p_n - k_i p_i = r_i$ and $p_i - r_i = (k_i+1)p_i - p_n$. But $p_n + (p_i - r_i)$ is divisible by p_i for $i = 1, \cdots, n-1$, where $r_1 \equiv 1$.

Now $2 + p_n = p_{n+1}$ if and only if $p_i - r_i \neq 2$ for any i. Note that $p_1 - r_1 \equiv 1$. But if $p_i - r_i = 2$ then $k_i + 1$ is odd, *i.e.*, $k_i = 2m_i$. Therefore $k_i + 1 \geqq 3$. But $p_{n+1} < 2p_n$ by Bertrand's conjecture. But the sieve (see the first three lines of the proof of I) only requires that we test whether a number is prime or not by dividing by those primes which are less than the square root of the number. Therefore we need test as to whether $k_i = 2m_i$ for those $p_i < \sqrt{(2p_n)}$.

Therefore the necessary and sufficient conditions that for a given p_n that $p_n + 2$ is a prime are as follows:

For each p_i such that

$$[p_n/p_i] = 2m_i \quad \textit{and} \quad 3 \leqq p_i < \sqrt{(2p_n)},$$

then $p_i - r_i > 2$, i.e., $(2m_i+1)p_i - p_n > 2$.

COROLLARY. *If $[p_n/p_i] = 2m_i + 1$ for $3 \leqq p_i < \sqrt{(2p_n)}$, then $p_n + 2$ is a prime.*

The hypothesis holds for $p_n = 3, 5, 11, 17$. It fails for p_n equal to the first of a prime pair up to and including the prime 3557. Since as n becomes larger the hypothesis fails for an i relatively small compared to n, it is my conjecture that there are no other prime pairs beyond 17, 19 that satisfy the hypotheses of this corollary.

III. Using congruences, the necessary conditions can be restricted to the following basic ones, which I have not found listed elsewhere. They can of course be expanded.

For $p_n > 3$:

$$p_n \equiv 1 \pmod 3 \quad \text{or} \quad p_n \equiv 2 \pmod 3.$$

But if $p_n + 2 = p_{n+1}$, then $p_n \not\equiv 1 \pmod 3$. Therefore

$$p_n \equiv 2 \pmod 3 \quad \text{or} \quad p_n = 3m + 2.$$

But since p_n is an odd number, then $3m$ is also odd. Therefore

$$p_n = 3(2r + 1) + 2 = 6r + 5. \qquad r = 0, 1, 2, \cdots$$

where $r \neq 5k,\ 5k+3,\ 7k,\ 7k+5$. $k = 0, 1, 2, \cdots$.

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THE ROLE OF INDUSTRIAL MEMBERS IN THE MATHEMATICAL ASSOCIATION OF AMERICA*

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Mathematical activity—like all of Gaul—may be divided into three major areas: Education, Research, and Applications. Each is of great importance, and in fact three major organizations, the Association, the Society, and SIAM, concentrate in these respective areas. And yet much of the strength of the mathematical fabric comes from the interaction among these three. The interaction between education and research has traditionally been provided by common personnel at the universities; on the high school level it has been weak for many years, but has recently been conspicuous in efforts such as the School Mathematics Study Group. The interaction between research and applications has become much stronger through the growth of computing and the increased use of mathematicians in industry, as well as through a number of new graduate programs in applied mathematics at the universities. My topic today, the role of the industrial member of the Association, is one aspect of the third of these interactions, that between applications and education.

When we consider the role of the industrial member of the Association, we can look at his activity from two points of view, that of the man and of the industry which employs him on the one hand, and that of the section and of the Association as a whole on the other. First of all, what will be the attitude of the industry? The company will undoubtedly be happy to have the man participate in any and all activities of the Association. Why? The crudest answer, of course, is publicity for the industry, pure and simple. But this is far from the whole truth. The best of industry has always felt its obligations to the universities, and it has become increasingly popular for companies to give scholarships, sponsor research contracts, and the like. Recently, however, it has become clear that money is not enough; industry has begun to contribute *people* back to education, both as visiting professors and lecturers, and as participants in the study and preparation of new curricula. Perhaps this is an indication of a somewhat troubled conscience, for industry drains a great many highly trained people from education. The best of industry also encourages its mathematicians to be mathematicians in the fullest sense of the word; if the mathematician feels that participation in the Association is part of his fulfillment, more power to him.

This brings us to the man himself. The same conscience operates, if anything more strongly, in him. He too feels that he should give something of himself back directly to the educational process which has prepared him. In addition, many mathematicians in industry, whether they admit it or not, miss teaching just a little bit, and greatly enjoy the opportunity to participate in many of the Association's activities.

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How does the Association benefit by this participation? Basically, it is as simple as this: Competent, dedicated, manpower is always welcome. There are many places where this is useful; not the least contribution to the activities of the section which the industrial mathematician can make is to give lectures at meetings. Novel applications of mathematics to, for example, economics, or communications, or military strategy, are fascinating, and make fine program material. One key point is that the depth of the mathematics required in interesting applications is frequently not so great but that the speaker can reach important results in a half hour without losing his audience along the way. Another point is that, because of his experience in explaining mathematics to his colleagues with less specialized training, the industrial mathematician should, despite his lack of teaching experience, command a quality of exposition which compares favorably to that of the average professor.

To the goal of the Association as a whole, the excellence of mathematics teaching, the industrial mathematician can also contribute in several ways. A large fraction of the people to whom mathematics is taught in college are, for instance, engineers who end up in industry. How good was their mathematical education? One way to find out is to ask the user, and the mathematician "on the spot" can often tell you a great deal about the mathematics which it is useful and important to know in his own organization. Finally, while no one knows exactly just why and how students learn mathematics, motivation from the physical world does play a part with some of them. By making known some of the exciting applications of mathematics in real situations, the industrial mathematician contributes to the enjoyment of mathematics by everyone.

A NOTE ON A GENERALIZATION OF BOOLEAN MATRIX THEORY

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1. Introduction. Let Q, $+$, $\cdot$ be an associative semiring [1], containing elements 0 and 1 with the following properties (for any $a \in Q$):

$$a + 0 = a, \qquad a \cdot 0 = 0 \cdot a = 0,$$
$$a \cdot 1 = 1 \cdot a = a, \qquad a + 1 = 1 + a = 1.$$

We shall call such a system Q, $+$, $\cdot$ a *Q-semiring*. Any distributive lattice L, $+$, $\cdot$ with 0 and 1, and especially any Boolean algebra obviously form examples of Q-semirings.

Another example, discussed by Shimbel [2] in connection with transportation and similar networks, is the following: Q is the set of nonnegative integers, together with $+\infty$; $a+b$ is defined as min (a, b), and $a \cdot b$ as the arithmetic sum

* On leave from Technion, Israel.

of a and b; $+\infty$ and the integer 0 take the role of the 0- and 1-elements, respectively.

This note discusses matrices over Q-semirings, generalizing results of Boolean and distributive lattice matrix theory [3], [4].

2. Some properties of Q-semirings. From the above definition of a Q-semiring the laws of absorption $a+a\cdot b=a$ and $a+b\cdot a=a$ follow immediately:

$$a + a\cdot b = a\cdot(1 + b) = a\cdot 1 = a,$$

$$a + b\cdot a = (1 + b)\cdot a = 1\cdot a = a.$$

Substituting 1 for b, we derive the law of additive idempotency: $a+a=a$. A Q-semiring thus forms a semilattice [5] with respect to addition, with 0 and 1 as zero and universal elements, respectively. It, therefore, forms a partially ordered system, $a\geqq b$ being defined by $a+b=a$. Evidently, $a\geqq b$ implies $xa\geqq xb$ and $ay\geqq by$ for all x, y. Repeated application of this rule leads to:

$$a_1\cdot\cdots\cdot a_k \geqq b_1\cdot a_1\cdot\cdots\cdot b_k\cdot a_k\cdot b_{k+1}. \tag{1}$$

3. Q-semiring matrices. Let Q, $+$, $\cdot$ be a Q-semiring, and Q_n the set of all $n\times n$ matrices $A=(a_{ij})$, $B=(b_{ij})$, $\cdots$, with elements in Q. In Q_n we define $A+B$, AB and $A\geqq B$ by:

$$A + B = (a_{ij} + b_{ij}), \qquad AB = \left(\sum_{k=1}^{n} a_{ik}b_{kj}\right),$$

$$A \geqq B \text{ if and only if } a_{ij} \geqq b_{ij} \text{ for every } i, j.$$

$A\geqq B$ evidently implies $A+B=A$ and conversely. One immediately verifies that Q_n, $+$ forms a semilattice with the matrix 0 (all entries 0) as zero element, the matrix E (all entries 1) as universal element, and with $A\geqq B$ as the corresponding partially ordering relation. With respect to addition and multiplication, Q_n forms an associative semiring, with the matrix I (all diagonal entries 1, all others 0) as multiplicative identity. Again, $A\geqq B$ implies $XA\geqq XB$ and $AY\geqq BY$ for all X, $Y\in Q_n$.

4. Q-nets. Hohn, Seshu, and Aufenkamp [6] have introduced the general concept of a net (*i.e.*, a weighted, directed graph) and have developed matrix methods for the analysis of such nets. In this paper, which is concerned with matrix theory, we introduce the concept of a "Q-net," essentially in order to facilitate the study of Q-semiring matrices. However, the results obtained form interesting generalizations of theorems on switching nets [3], [4] and transportation networks [2].

A *Q-net* is defined as a directed graph consisting of n vertices $v_1, \cdots, v_n$, with exactly one branch b_{ij}, $i\neq j$, from each v_i to each other v_j, each branch b_{ij} being weighted by an element w_{ij} of a given Q-semiring. The matrix $C=(c_{ij})$ defined by

$$c_{ij} = \begin{cases} 1 & \text{if } i = j, \\ w_{ij} & \text{if } i \neq j, \end{cases}$$

is called the *connection matrix* of the Q-net.

A *directed path* of a Q-net of length r from v_i to v_j is defined as a sequence of r branches of the form $b_{ik_1}, b_{k_1k_2}, \cdots, b_{k_{r-1}j}$. If $i=j$ the path is called *closed*, otherwise *open*. If $i, k_1, \cdots, k_{r-1}, j$ are all distinct, the path is called *proper*. An open path, which is not proper, is called *redundant*. The *weight* w of a directed path is defined as the product of its branch weights:

$$w = w_{ik_1} \cdot w_{k_1k_2} \cdot \cdots \cdot w_{k_{r-1}j}.$$

Given any two distinct vertices v_i and v_j, $i \neq j$, of a Q-net, the *tranmission* t_{ij} from v_i to v_j is defined as the sum of the weights of all directed paths from v_i to v_j. The *transmission matrix* T of a Q-net is defined by $T=(t_{ij})$, where $t_{ii}=1$, $i=1, \cdots, n$, and t_{ij}, $i \neq j$, is the above defined transmission.

THEOREM 1. *Let C and T be the connection and transmission matrices, respectively, of a Q-net with n vertices. Then*

$$C^m = T \text{ for any } m \geqq n-1. \tag{2}$$

The proof of this theorem is based on the following lemmas of which Lemma 1 is evident.

LEMMA 1. *The maximal length of any proper path is $n-1$.*

LEMMA 2. *Let w be the weight of an open redundant path p from v_i to v_j. Then there exists a proper path from v_i to v_j with weight w', such that $w' \geqq w$.*

Proof. A proper path p' from v_i to v_j may be obtained from the original redundant path p by eliminating some of its branches. Applying (1) to weights w' and w of paths p' and p we obtain $w' \geqq w$.

LEMMA 3. *The (i, j)-entry, $i \neq j$, of C^m, $m \geqq 1$, is the sum of the weights of all directed paths from v_i to v_j with length $r \leqq m$.*

Proof. The (i, j)-entry of C^m is

$$c_{ij}^{(m)} = \sum_{(k)} c_{ik_1} c_{k_1k_2} \cdots c_{k_{m-1}j}.$$

The weight of any directed path of length m will be a term of this sum. Furthermore, $C \geqq I$ implies $C^m \geqq C^r$, *i.e.*, $c_{ij}^{(m)} \geqq c_{ij}^{(r)}$, for any $r \leqq m$. It follows that $c_{ij}^{(m)}$ includes the weight of any path of length $r \leqq m$ from v_i to v_j. Conversely, any term of $c_{ij}^{(m)}$ represents the weight of a directed path of length $r \leqq m$ from v_i to v_j. Lemma 3 is thus proved.

Proof of Theorem 1. By Lemma 3 and the definition of T, $C^m \leqq T$, for any $m \geqq 1$. Now let p be a directed path from v_i to v_j, $i \neq j$, of length $r > n-1$ and weight

w. Then, by Lemma 1, p is redundant and by Lemma 2 there exists a proper path p' from v_i to v_j with weight $w' \geqq w$. By Lemma 1, p' is of length $r' \leqq n-1$. Hence, by Lemma 3, $T = C^{n-1}$.

Furthermore, $C \geqq I$, implies $C^m \geqq C^{n-1}$, for any $m \geqq n-1$. We thus have, for any $m \geqq n-1$, both $C^m \leqq T$ and $C^m \geqq C^{n-1} = T$, whence (2).

Theorem 1 generalizes similar results obtained in [2], [3], and [4]. A restricted form of this Theorem was first stated by A. G. Lunts [7].

5. Matrix theorems. We shall now use Theorem 1 to prove the following theorem on Q-semiring matrices:

THEOREM 2. *Let A be an $n \times n$ matrix over a Q-semiring with $A \geqq I$. Then $I \leqq A \leqq \cdots \leqq A^{n-1} = A^n = A^{n+1} = \cdots$.*

Proof. $A \geqq I$ implies $A^k \geqq A^{k-1}$. Because of $A \geqq I$, we may consider A as the connection matrix of a Q-net. The rest of Theorem 2 therefore follows immediately from Theorem 1.

The corresponding theorem on Boolean matrices is due to Lunts [7] (see also [3], Th. 3.2.1). Introducing the notation

$$A^* = \sum_{k=1}^{\infty} A^k,$$

we obviously have: $A \geqq I$ implies $A^* = A^{n-1}$.

THEOREM 3. *Let A and B be $n \times n$ matrices over a Q-semiring with $A \geqq I$, $B \geqq I$. Then*

$$(A + B)^* = (A^*B^*)^* = (B^*A^*)^*. \tag{3}$$

Proof. $A^* \geqq A$ and $B^* \geqq I$ implies $A^*B^* \geqq A$. Similarly $A^*B^* \geqq B$. Therefore $A^*B^* \geqq A + B$; whence

$$(A^*B^*)^* \geqq (A + B)^*. \tag{4}$$

Now $A^* \leqq (A+B)^*$ and $B^* \leqq (A+B)^*$. Therefore $A^*B^* \leqq ((A+B)^*)^2 = (A+B)^*$; whence

$$(A^*B^*)^* \leqq ((A + B)^*)^* = (A + B)^*. \tag{5}$$

Combining (4) and (5) we obtain (3). Theorem 3 is a modification of an analogous theorem of the algebra of relations [8].

6. Determinants and adjoints. We shall now restrict ourselves to commutative Q-semirings. Let A be an $n \times n$ matrix over a commutative Q-semiring. We define its determinant $|A|$ by

$$|A| = \sum a_{1h_1} \cdots a_{nh_n},$$

where the summation is taken over all permutations $(h_1, \cdots, h_n)$ of $(1, \cdots, n)$.

This definition, coinciding with the definition of permanent in ordinary matrix theory [9], generalizes the concept of Boolean determinant [10]. One easily verifies that the usual expansion methods of determinants also apply to the above definition, just as they apply to permanents in ordinary matrix theory.

The adjoint $\hat{A}$ of A is defined by $\hat{A}=(A_{ij})$, where A_{ij} is the cofactor of a_{ji} in $|A|$.

THEOREM 4. *Let A be an $n\times n$ matrix over a commutative Q-semiring with $A\geqq I$. Then $\hat{A}=A^*$.*

Proof. We consider the Q-net having A as its connection matrix. We then have to prove that the transmission matrix $T=(t_{ij})$ of the net is given by $T=\hat{A}$, *i.e.*, $A_{ii}=1$, and the transmission t_{ij} from v_i to v_j, $i\neq j$, equals A_{ij}. That $A_{ii}=1$ follows immediately from $a_{ii}=1$. Now, consider any pair (i, j), $i, j=1, \cdots, n$, with $i\neq j$.

Let H be the set of permutations

$$h=\begin{pmatrix}1 & \cdots & n\\ h_1 & \cdots & h_n\end{pmatrix}$$

with $h_j=i$. Let $h=u\cdot v\cdot \cdots$ be the decomposition of h into cyclic permutations, u having the form $u=(j, i, \cdots, j)$. Let U denote the set of all such permutations u. Writing A_h instead of $a_{1h_1}\cdots a_{nh_n}$, we have $A_h=A_u\cdot A_v\cdots$; therefore $A_u\geqq A_h$. Because $H\supseteq U$, we have

$$\sum_{h\in H} A_h = \sum_{u\in U} A_u.$$

Evidently, $\sum_{u\in U} A_u=a_{ji}t_{ij}$, whereas $\sum_{h\in H} A_h$ consists of all terms of $|A|$ containing a_{ji}. t_{ij} is, therefore, the cofactor of a_{ji} in $|A|$, *i.e.*, $t_{ij}=A_{ij}$; whence $T=\hat{A}$. Theorem 4 is thus proved. Furthermore, we have also covered the proof of the following:

COROLLARY. *Let C and T be the connection and transmission matrices of a Q-net. Then $T=\hat{C}$.*

This corollary is a generalization of a result due to Aranovich [10]. Its applicability to transportation networks, for example, is noteworthy.

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MATHEMATICAL NOTES

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NOTE ON COMPLETE SEQUENCES OF INTEGERS

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Recently, Hoggatt and King [1] have defined an arbitrary sequence, $\{f_i\}_{i=1}^{\infty}$, of positive integers to be complete if, and only if, every positive integer n can be represented in the form $n=\sum_1^{\infty}\alpha_i f_i$, where each α_i is either zero or unity. Here the α_i are determined by n, although, in general, this determination is not unique; further, for each n, there exists an $N(n)$ such that $\alpha_i=0$ for $i>N(n)$.

The purpose of this note is to give a simple necessary and sufficient condition for the completeness of such number sequences and to show that the Fibonacci numbers are characterized by certain properties involving completeness. In the following, α_i and β_i will always denote quantities which are either zero or unity for each value of the subscript, and n will represent an arbitrary positive integer.

LEMMA. *Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of positive integers (not necessarily distinct) with $f_1=1$ and satisfying $f_{p+1}\leqq 1+\sum_1^p f_i$ for $p=1, 2, \cdots$. Then for $0<n<1+\sum_1^k f_i$, there exist $\{\alpha_i\}_{i=1}^{k}$ such that $n=\sum_1^k \alpha_i f_i$.*

Proof. The lemma obviously holds for $k=1$; assume that it holds for $k=N$. Then we must show that $0<n<1+\sum_1^{N+1} f_i$ implies the existence of $\{\beta_i\}_{i=1}^{N+1}$ such that $n=\sum_1^{N+1}\beta_i f_i$. We need only consider values of n satisfying $1+\sum_1^N f_i \leqq n<1+\sum_1^{N+1} f_i$ since the case $n<1+\sum_1^N f_i$ is covered by the induction hypothesis. Thus, $n-f_{N+1}\geqq 1+\sum_1^N f_i-f_{N+1}\geqq 0$ by assumption. If $n-f_{N+1}=0$, the conclusion follows; otherwise, $0<n-f_{N+1}<1+\sum_1^N f_i$ implies the existence of $\{\alpha_i\}_{i=1}^{N}$ such that $n-f_{N+1}=\sum_1^N \alpha_i f_i$. The lemma is immediate on transposing f_{N+1} and identifying $\beta_i=\alpha_i$ for $i=1, \cdots, N$ and $\beta_{N+1}=1$.

THEOREM 1. *Let* $\{f_i\}_{i=1}^{\infty}$ *be a nondecreasing sequence of positive integers with* $f_1=1$. *Then* $\{f_i\}$ *is complete if and only if* $f_{p+1} \leqq 1+\sum_1^p f_i$ *for* $p=1, 2, \cdots$.

Proof. The sufficiency follows directly from the lemma. To show necessity, assume there exists $n_0 \geqq 1$ such that $f_{n_0+1} > 1+\sum_1^{n_0} f_i$. Then $f_{n_0+1} > f_{n_0+1}-1 > \sum_1^{n_0} f_i$ which implies that the positive integer $f_{n_0+1}-1$ cannot be represented in the form $\sum_1^{\infty} \beta_i f_i$.

Example. For the Fibonacci sequence, $F_1 = 1$, $F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ $(n=1, 2, \cdots)$, completeness follows immediately from the well-known result [2], $F_{p+1} < F_{p+2} = 1+\sum_1^p F_i$ $(p=1, 2, \cdots)$.

COROLLARY. *If* $\{f_i\}$ *is an arbitrary nondecreasing complete sequence of positive integers, then* $f_i \leqq \phi_i$ $(i=1, 2, \cdots)$ *where* $\phi_i = 2^{i-1}$.

Proof. The corollary obviously holds for $i=1$. Assuming it for $i=n$, we have $f_{n+1} \leqq 1+\sum_1^n f_i \leqq 1+\sum_1^n \phi_i = \phi_{n+1}$, which proves the assertion.

From the relation $\phi_{n+1} = 1+\sum_1^n \phi_i$ it is seen that $\{\phi_i\}$ is a complete sequence; the preceding corollary then shows that $\{\phi_i\}$ is the fastest growing complete sequence in the sense that any other nondecreasing complete sequence is dominated by the $\{\phi_i\}$ termwise. The representation $n = \sum_1^{k(n)} \alpha_i \phi_i$ afforded by $\{\phi_i\}$ for an arbitrary integer n is such that $\alpha_k \alpha_{k-1} \cdots \alpha_1$ is the binary representation of n. As a consequence, representation in terms of the $\{\phi_i\}$ is unique. Furthermore, this property characterizes the sequence $\{\phi_i\}$. For let $\{f_i\}$ be a nondecreasing complete sequence of positive integers such that $\{f_i\}$ is not identical to the sequence $\{\phi_i\}$ and let k be the smallest integer such that $f_k \neq \phi_k$. By Theorem 1, $f_k \leqq 1+\sum_1^{k-1} f_i = 1+\sum_1^{k-1} \phi_i = \phi_k$.

Since $f_k \neq \phi_k$, we must have $f_k < 1+\sum_1^{k-1} f_i$ and hence $f_k = \sum_1^{k-1} \alpha_i f_i$ by the lemma. But the integer f_k can also be represented by the single term f_k itself and thus has two distinct representations in terms of the $\{f_i\}$.

It is of interest [1] to determine when the completeness of a sequence is left unaltered by the removal of an arbitrary term of the sequence.

THEOREM 2. *Let* $\{f_i\}$ *be a nondecreasing complete sequence of positive integers. The condition,* $f_{p+1} \leqq 1+\sum_1^{p-1} f_i$ $(p=1, 2, \cdots)$, *is necessary and sufficient for the sequence to remain complete after the deletion of an arbitrary term.*

Proof. Straightforward by application of Theorem 1 to the deleted sequence.

Example. Since $F_{p+1} = 1+\sum_1^{p-1} F_i$ for the Fibonacci sequence, this sequence retains the completeness property after the deletion of an arbitrary term.

THEOREM 3. *Let* $\{f_i\}$ *be a nondecreasing complete sequence of positive integers with* $f_{p+1} \geqq 1+\sum_1^{p-1} f_i$ $(p=1, 2, \cdots)$. *Then the deletion of two arbitrary terms, say* f_μ *and* $f_\nu (\mu \neq \nu)$, *is sufficient to destroy completeness.*

Proof. Assume without loss of generality that $\mu < \nu$. The case $\mu=1$, $\nu=2$ obviously destroys completeness since every complete sequence of positive integers

must contain a term equal to unity. For $\nu>2$, $f_{\nu+1}\geqq 1+\sum_1^{\nu-1} f_i=1+\sum_1^{\mu-1} f_i + \sum_{\mu+1}^{\nu-1} f_i+f_\mu>1+\sum_1^{\mu-1} f_i+\sum_{\mu+1}^{\nu-1} f_i$. But this violates the condition of Theorem 1 for completeness which requires that each term of the deleted sequence be less than or equal to one plus the sum of all preceding terms.

Note that the Fibonacci sequence $\{F_i\}$ satisfies the hypotheses of Theorem 3 and, therefore, possesses the stated property.

The following partial converse to Theorem 3 provides the basis for a characterization of the Fibonacci numbers.

THEOREM 4. *Let $\{f_i\}_{i=1}^{\infty}$ be a complete sequence of positive integers such that* (i) *deletion of an arbitrary term does not affect completeness; and* (ii) $f_1=f_2=1$ *and* $f_{i+1}>f_i$ *for* $i\geqq 2$.

Then, the deletion of two arbitrary terms destroys completeness if, and only if,

$$(*) \qquad f_{p+1} \geqq 1+\sum_1^{p-1} f_i \qquad (p=1,2,\cdots).$$

Proof. Condition (*) is sufficient by Theorem 3. To show necessity, assume that the deletion of any two terms of the sequence renders the deleted sequence incomplete and that (*) is not satisfied. Then there exists $n_0\geqq 2$ such that $f_{n_0+1}<1+\sum_1^{n_0-1} f_i$. Now, consider the sequence which results from the original sequence after deleting the specific terms f_{n_0} and $f_1=1$. For $p<n_0$,

$$(1) \qquad f_{p+1} \leqq 1+\sum_2^{p} f_i \qquad \text{(by (i) and Theorem 1).}$$

$$(2) \qquad f_{n_0+1}<1+\sum_1^{n_0-1} f_i \leqq 1+\sum_2^{n_0-1} f_i, \quad \text{since} \quad f_1=1.$$

For $p>n_0$,

$$f_{p+1} \leqq 1+\sum_1^{n_0-1} f_i + f_{n_0} + \sum_{n_0+1}^{p-1} f_i \qquad \text{(by (i) and Theorem 2).}$$

Thus,

$$f_{p+1}<1+\sum_1^{n_0-1} f_i+\sum_{n_0+1}^{p} f_i \qquad \text{(by (ii))}$$

and

$$(3) \qquad f_{p+1} \leqq 1+\sum_2^{n_0-1} f_i+\sum_{n_0-1}^{p} f_i \cdot (p>n_0).$$

But inequalities (1), (2), and (3) together with Theorem 1 imply that the deleted sequence is complete, in contradiction to our assumption.

Lastly, we give a characterization of the Fibonacci sequence:

THEOREM 5. *Let* $\{f_i\}_{i=1}^{\infty}$ *be an arbitrary complete sequence of positive integers having at most two elements equal and possessing the following properties:*

(*A*) *deletion of an arbitrary member of the sequence does not destroy completeness.*

(*B*) *deletion of any two members of the sequence renders the deleted sequence incomplete.*

Then if $\{g_i\}_{i=1}^{\infty}$ *denotes the sequence obtained by rearranging the terms of* $\{f_i\}$ *in nondecreasing order,* $g_i = F_i$ *for* $i = 1, 2, \cdots$, *where* $\{F_i\}$ *is the Fibonacci sequence.*

Proof. Since the stated properties are invariant with respect to reordering, the sequence $\{g_i\}_{i=0}^{\infty}$ satisfies both (A) and (B). By Theorem 2, $g_{p+1} \leqq 1 + \sum_1^{p-1} g_i$ $(p = 1, 2, \cdots)$ while by Theorem 4, $g_{p+1} \geqq 1 + \sum_1^{p-1} g_i$ $(p = 1, 2, \cdots)$. Thus

$$g_{p+1} = 1 + \sum_1^{p-1} g_i \qquad \text{for } p = 1, 2, \cdots \tag{4}$$

and $g_1 = 1$, since every complete sequence must contain unity. (In addition, $g_2 = 1$, and because of our assumption that the sequence has at most two identical terms, the g_i are distinct and hence increasing for $i \geqq 2$.) Noting that (4) is the recurrence relation satisfied by the Fibonacci numbers, we have $g_i = F_i$ for every $i \geqq 1$.

Certain properties related to completeness have been analyzed by Daykin [3] in connection with the introduction and study of generalized Fibonacci numbers. Unique representation of an arbitrary integer in terms of these generalized numbers is possible if the admissible form of the representation is suitably constrained; moreover, the generalized Fibonacci sequences are the only sequences possessing this property.

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2. S. E. Ganis, Notes on the Fibonacci sequence, this MONTHLY, vol. 66, 1959, pp. 129–130.

3. D. E. Daykin, Representation of natural numbers as sums of generalized Fibonacci numbers, J. London Math. Soc. vol. 35, 1960, pp. 143–160.

A DIVISIBILITY PROPERTY OF THE BINOMIAL COEFFICIENTS*

L. CARLITZ, Duke University

Let p be prime. It is familiar that the binomial coefficients $\binom{p}{r}$ $(0 < r < p)$ are all divisible by p. It is perhaps less familiar that, for fixed n, the binomial coefficients $\binom{n}{r}$ $(0 < r < n)$ are all divisible by p if and only if n is a power of p.

Let

$$n = n_0 + n_1 p + \cdots + n_k p^k \qquad (0 \leqq n_j < p) \tag{1}$$

and put $S(n) = n_0 + n_1 + \cdots + n_k$. Schäffer ([3], Lemma 3) has proved that the

* Research supported by National Science Foundation Grant NSF-G 9425.

binomial coefficients $\binom{n}{(p-1)r}(0<(p-1)r<n)$ are all divisible by p if and only if $S(n)\leqq p-1$.

We shall prove the following theorem which includes both of these results.

THEOREM. *Let p be a fixed prime and k a fixed integer such that $k\mid p-1$. Then the binomial coefficients $\binom{n}{kr}(0<kr<n)$ are all divisible by p if and only if*

$$S(n) \leqq k. \tag{2}$$

Proof. Lucas has proved ([1], p. 52) that

$$\binom{n}{r} \equiv \binom{n_0}{r_0}\binom{n_1}{r_1}\cdots\binom{n_k}{r_k} \pmod{p}, \tag{3}$$

where n_j is defined by (1) and

$$r = r_0 + r_1 p + \cdots + r_k p^k \qquad (0 \leqq r_j < p). \tag{4}$$

If $k\mid r$ then by (4), $k\mid r_0+r_1+\cdots+r_k$, so that (since $r>0$), $r_0+r_1+\cdots+r_k\geqq k$. Assume that n satisfies (2), so that $n_0+n_1+\cdots+n_k\leqq k$. Thus

$$n_0 + n_1 + \cdots + n_k \leqq r_0 + r_1 + \cdots + r_k. \tag{5}$$

It follows from (5) that $r_j\geqq n_j$ for at least one j and, indeed, $r_j>n_j$ for at least one j unless $n=r$. But by (3), $r_j>n_j$ implies $\binom{n}{r}\equiv 0 \pmod{p}$. This evidently proves the sufficiency of the condition (2).

Next assume that $S(n)>k$. Then we can find nonnegative integers r_0, r_1, $\cdots$, r_k such that $r_j\leqq n_j$ $(j=0, 1, \cdots k)$ and $r_0+r_1+\cdots+r_k=k$. If we put $r=r_0+r_1p+r_2p^2+\cdots+r_kp^k$ it follows that $k\mid r$, $0<r<n$ and $\binom{n}{r}\not\equiv 0 \pmod{p}$. This completes the proof of the theorem.

The condition $k\mid p-1$ is evidently necessary. For example, if $p=5$, $k=3$, and $n=7$, we have $S(7)=3$ but $\binom{7}{6}=7\not\equiv 0 \pmod 5$.

We remark that McCarthy [2] has applied Schäffer's lemma to determine the Bernoulli polynomials $B_n(x)$ such that $pB_n(x)$ are Eisenstein polynomials with determining prime p. References to other results on divisibility properties of binomial and multinomial coefficients will be found in L. E. Dickson's *History of the Theory of Numbers*, vol. 1, Ch. IX. Washington, 1919.

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1. E. Lucas, Sur les congruences des nombres euleriennes et des coefficients differentiels des fonctions trigonométriques, Bull. Soc. Math. France, vol. 6, 1878, pp. 49–54.

2. P. J. McCarthy, Irreducibility of certain Bernoulli polynomials, this MONTHLY, vol. 68, 1961, pp. 352–353.

3. J. J. Schäffer, The equation $1^p+2^p+3^p\cdots+n^p=m^q$, Acta Math. vol. 95, 1956, pp. 155–189.

ON THE PERIODICITY OF HOMEOMORPHISMS OF THE REAL LINE

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It is interesting to note that a complete description of the periodic homeomorphisms of the real line can be derived using only the simplest properties of monotone continuous functions.

DEFINITION. *A homeomorphism $f(\cdot)$ is periodic of period n if $f(\cdot)$ composed with itself n times $[f(f(\cdots(f(\cdot))\cdots))$ with n terms$]$ is the identity function and n is the least such integer.*

LEMMA. *Every continuous $1-1$ map of the real line onto itself is either order-preserving or order-reversing.*

Proof. Let $r<s<t$ and $f(r)>f(s)<f(t)$ where $f(\cdot)$ is a continuous $1-1$ map of the real line onto itself. By the intermediate value theorem (see *Advanced Calculus* by R. C. Buck, New York, 1956, page 41, Theorem 12) for any u such that $f(s)<u<\min(f(r),f(t))$ there exist v and w, $r<v<s$ and $s<w<t$, such that $f(v)=f(w)=u$. Since $f(\cdot)$ is $1-1$ this is impossible. Similarly $r<s<t$ and $f(r)<f(s)>f(t)$ is impossible. Thus $f(\cdot)$ is either order-preserving or order-reversing.

COROLLARY. *Every continuous $1-1$ map of the real line onto itself is a homeomorphism.*

THEOREM. *Let $f(\cdot)$ be any homeomorphism of the real line such that $f(\cdot)$ composed with itself n times, n a positive integer, is the identity function. If $f(\cdot)$ is order-preserving it is the identity function. If $f(\cdot)$ is order-reversing then its period is two and $f(f(x))=x$ for all real x.*

Proof. If $f(\cdot)$ is order-preserving and if there exists x such that $f(x)>x$, then $f(f(x))>f(x)>x$. By induction the nth iterate $f^{(n)}(x)>x$ for every positive integer n. Thus $f(\cdot)$ can not be periodic under composition. Similarly $f(y)<y$ for some y implies $f(\cdot)$ not periodic. Every periodic homeomorphism other than the identity function is thus order-reversing.

Let $f(\cdot)$ be order-reversing and periodic of period n; $f(f(\cdot))$ is also periodic with period either n or $n/2$. The composition of two order-reversing functions is order-preserving. Thus, by the first paragraph, $f(f(\cdot))$ is the identity function.

There exist homeomorphisms of period two and a method for generating them. A simple use of the intermediate value theorem shows that every homeomorphism of period two has exactly one fixed point. Therefore every such homeomorphism $f(\cdot)$ can be generated by choosing for some point b a function $g(\cdot)$ defined for all $x\geqq b$, $1-1$, monotone decreasing, continuous, and satisfying $g(b)=b$ and $\lim_{x\to\infty} g(x)=-\infty$. Then the function $f(\cdot)$ defined by $f(x)=g(x)$ for $x\geqq b$ and $f(x)=g^{-1}(x)$ for $x<b$ is a homeomorphism of period two. If $f(\cdot)$ rather than $g(\cdot)$ is originally specified, $g(\cdot)$ can be defined readily. Let b be the fixed

* It is with deep regret that the Editor has learned of the death of Neill McShane in an automobile accident, March 11, 1961.

point of $f(\cdot)$ and let $g(x)=f(x)$ for $x\geqq b$.

Examples of homeomorphisms of period two are the functions $f(\cdot)$, $g(\cdot)$, and $h(\cdot)$, where $f(x)=b-x$ for all real x and some real constant b; $g(x)=-1+\ln(-x)$ for $x<-1$ and $-e^{x+1}$ for $x\geqq -1$; and $h(x)=-x^n$ for $x\geqq 0$ and $(-x)^{1/n}$ for $x<0$. (Here n is a *positive* constant.)

CORRECTION

A. K. Rajagopal, *On some of the classical orthogonal polynomials*, this MONTHLY, vol. 67, 1960, pp. 166–169. Equation (3) should be $(d^n/dx^n)(\exp U)=(\exp U)(d/dx)+U')^n\cdot 1$.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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WHEN DOES ZERO CORRELATION IMPLY INDEPENDENCE?

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1. Statistics textbooks point out that zero correlation implies independence for normal distributions, but not for distributions in general. There is a simple class G of bivariate distributions for which the subclass M with this property can be explicitly determined, where M properly contains the bivariate normals.

2. Let G be the bivariate distributions with densities of the form

$$g(x,y)=r(x\cdot y)\exp\left(-\frac{x^2}{2\sigma_1^2}-\frac{y^2}{2\sigma_2^2}\right),\qquad -\infty<x,y<\infty, \tag{1}$$

where the "linking function" r has the property that $r(u)-r(-u)$ does not change sign for $u\in[0,\infty)$.

Let I, Z, M, N_ρ be subsets of G defined respectively by

I: $r(u)=C|u|^\alpha$, $\quad\alpha>-1$;

Z: $r(u)=r(-u)$ almost everywhere;

M: $M=(G-Z)\cup I=G-(Z-I)$;

N_ρ: $r(u)=(1-\rho^2)^{1/2}/(2\pi d)\exp\rho u/d$, $\quad d=\sigma_1\sigma_2$, $\quad|\rho|<1$.

Clearly the distributions in I are "independent" (that is, products of independent univariate distributions). The converse is readily proved, for if r belongs to an "independent" distribution, then as is easily shown $r(xy)=\text{const.}\ r(x)r(y)$, whose only positive measurable solutions are of the form $r(u)=C|u|^\alpha$. The con-

dition $\alpha > -1$ comes from the integrability of the density. These yield χ distributions with $\alpha+1$ degrees of freedom.

Since the marginal densities of distributions in G are even functions, the marginal first moments are zero if they exist. Thus the covariance of a distribution in G is, if it exists,

$$\sigma_{12} = \iint xyr(xy) \exp\left(-\frac{x^2}{2\sigma_1^2} - \frac{y^2}{2\sigma_2^2}\right) dxdy$$

$$= 2\int_0^\infty \int_0^\infty xy(r(xy) - r(-xy)) \exp\left(-\frac{x^2}{2\sigma_1^2} - \frac{y^2}{2\sigma_2^2}\right) dxdy.$$

Since $r(u)-r(-u)$ has one sign for $u \in [0, \infty)$, $\sigma_{12}=0$ if and only if $r(u) = r(-u)$ almost everywhere. Z, then, is the set of uncorrelated distributions, and properly contains I.

The distributions in $M=(G-Z)\cup I$ have the property that zero correlation implies independence, since the only even linking functions for these distributions have the independent form.

3. Clearly N_ρ is the set of normal distributions with correlation ρ, and since for these distributions

$$r(u) - r(-u) = (1 - \rho^2)^{1/2}/(\pi d) \sinh \rho u/d,$$

it follows that $N_\rho \subset G$.

If $N = \bigcup_\rho N_\rho$, over $|\rho| < 1$, then $N \cap Z = N \cap I = N_0$, and therefore the set N of bivariate normal distributions is a proper subset of M.

4. It is easy to define a class P of densities with the desired property, yet disjoint from N. Let s be a noneven linking function for a density in $G-Z$, take $\alpha \neq 0$, and choose a and b to make $r(u) = a|u|^\alpha + bs(u)a$ linking function. Let P be the corresponding densities. Note that $P \subset G$, $P \cap Z = P \cap I$, $P \subset M$, $P \cap N = 0$.

5. A necessary and sufficient condition for independence, derived by Lancaster [1] and Sarmatov [2], and cited by the reviewer of this note, is the vanishing of generalized correlation coefficients defined in terms of multi-orthonormal functions on the marginal distributions. This completely resolves the relationship between independence and zero-correlation, but probably requires graduate-level analysis for understanding.

6. The construction of G was prompted by a pedagogical discussion with Professor R. F. Tate, University of Washington, Seattle.

References

1. H. O. Lancaster, Zero correlation and independence. Austral. J. Statist., vol. 1, 1959, pp. 53–56.

2. O. V. Sarmatov, The maximum correlation coefficient (The asymmetrical case), Dokl. Akad. Nauk SSSR., vol. 121, 1958, pp. 52–55.

$\int \sec\theta\, d\theta$

NORMAN SCHAUMBERGER, Bronx Community College

The integral of sec θ, which is usually handled by multiplying numerator and denominator by sec $\theta+\tan\theta$, can be treated in the following interesting manner (suggested by Jesse Douglas of The City College of New York).

The student is aware that the sec θ and tan θ are closely related. This might suggest that we take differentials of each,

$$d(\sec\theta) = \sec\theta\tan\theta d\theta, \qquad d(\tan\theta) = \sec^2\theta d\theta.$$

If we now add the two expressions and factor the right side we obtain $d(\sec\theta+\tan\theta) = (\sec\theta+\tan\theta)\sec\theta d\theta$. Dividing by sec $\theta+\tan\theta$ and integrating, we have the desired result.

ON THE EULER—CAUCHY EQUATION

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One of the standard examples appearing in almost every text on differential equations is the so-called Euler-Cauchy equation,

$$(1) \qquad a_n x^n \frac{d^n y}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 x\frac{dy}{dx} + a_0 y = h(x).$$

This equation is solved by showing that "the change of variable, $x=e^t$" reduces it to a linear differential equation with constant coefficients. The procedure involves some juggling of derivatives and differentials which, although easily followed, is hardly understood (even by the better students) and certainly, as far as clarity is concerned, leaves much to be desired. This need not be the case. In this note we will show how, when looked at in the proper light, the procedure whereby (1) is solved—which is in fact nothing more than a trivial, but particularly nice, consequence of the rule for the differentiation of composite functions—can be presented in a straightforward and transparent manner. We do this *à la* Menger.

LEMMA.* *If* $g=f\,exp$, *then* $exp^n\cdot \mathbf{D}^n f\,exp = \mathbf{D}(\mathbf{D}-1)\cdots(\mathbf{D}-\mathbf{n}+1)g$, *for* $n=1, 2, \cdots$.

Proof. We proceed by induction. First of all,

$$\mathbf{D}g = exp\cdot \mathbf{D}f\,exp.$$

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* Substitution of functions is denoted by juxtaposition, and multiplication by a dot. Thus $g=f\,exp$ means $g(x)=f(exp\,x)=f(e^x)$, for any number x; $\mathbf{D}^n f\,exp$ is the function whose value at x is $\mathbf{D}^n f\,(exp\,x)$, or $f^{(n)}(e^x)$; j is the identity function and j^n is the nth power function, *i.e.*, $j(x)=x$ and $j^n(x)=x^n$; and $\mathbf{D}^0 f = f$. Note also that $hj=jh=h$, for any function h.

We also adhere to the typographical convention of denoting operators in boldface, functions in italic, and numbers in roman type.

Now suppose that for some integer k,

$$exp^k \cdot \mathbf{D}^k f\, exp = \mathbf{D}(\mathbf{D}-1)\cdots(\mathbf{D}-\mathbf{k}+1)g.$$

Then, differentiating, we have,

$$k \cdot exp^k \cdot \mathbf{D}^k f\, exp + exp^{k+1} \cdot \mathbf{D}^{k+1} f\, exp = \mathbf{D}^2(\mathbf{D}-1)\cdots(\mathbf{D}-\mathbf{k}+1)g;$$

whence

$$\begin{aligned} exp^{k+1} \cdot \mathbf{D}^{k+1} f\, exp &= [\mathbf{D}^2(\mathbf{D}-1)\cdots(\mathbf{D}-\mathbf{k}+1) - \mathbf{k}\mathbf{D}(\mathbf{D}-1)\cdots(\mathbf{D}-\mathbf{k}+1)]g \\ &= \mathbf{D}(\mathbf{D}-1)\cdots(\mathbf{D}-\mathbf{k})g. \end{aligned}$$

THEOREM. *If f satisfies the linear differential equation,*

$$\sum_{k=0}^{n} a_k \cdot j^k \cdot \mathbf{D}^k f = h, \tag{2}$$

where $a_0, a_1, \cdots, a_n$ are constant functions, then $g = f\, exp$ satisfies the differential equation $\mathbf{L}g = h\, exp$, where **L** *is the linear differential operator with constant coefficients,*

$$\mathbf{a}_0\mathbf{D}^0 + \sum_{k=1}^{n} \mathbf{a}_k[\mathbf{D}(\mathbf{D}-1)\cdots(\mathbf{D}-\mathbf{k}+1)].$$

Proof. Substituting *exp* into both sides of (2) yields,

$$\sum_{k=0}^{n} a_k \cdot exp^k \cdot \mathbf{D}^k f\, exp = h\, exp$$

which, on applying the lemma to the left-hand side, reduces to the desired result.

COROLLARY. *If g is a solution of $\mathbf{L}g = h\, exp$, then $f = g\, log$ is a solution of* (2).

By replacing the exponential function with other functions, this technique of solving the Euler-Cauchy equation may be applied in numerous other cases. For example, let $g = f\, sin$, so that

$$\begin{aligned} \mathbf{D}g &= \mathbf{D}f\, sin \cdot cos, \\ \mathbf{D}^2 g &= \mathbf{D}^2 f\, sin \cdot cos^2 - \mathbf{D}f\, sin \cdot sin \\ &= \mathbf{D}^2 f\, sin \cdot (1 - sin^2) - \mathbf{D}f\, sin \cdot sin. \end{aligned}$$

In this case we find that if f satisfies the differential equation,

$$(1 - j^2) \cdot \mathbf{D}^2 f - j \cdot \mathbf{D}f + f = h$$

then $g = f\, sin$ satisfies

$$\mathbf{D}^2 g + g = h\, sin.$$

Since *sin* and *cos* are two independent solutions of $\mathbf{D}^2g+g=0$, it follows that *sin arcsin* and *cos arcsin*, *i.e.*, j and $\sqrt{(1-j^2)}$, are two independent solutions of $(1-j^2)\cdot\mathbf{D}^2f-j\cdot\mathbf{D}f+f=0$.

Further examples can be constructed at will.

A FURTHER NOTE ON δ AND ϵ

ALLAN DAVIS, University of Utah

In Classroom Notes (*A note on* δ *and* ϵ, this MONTHLY, vol. 67, p. 780), Professor A. H. Sprague is concerned with the instructive exercise of finding, for given $\epsilon>0$, a corresponding $\delta>0$ in order to show from the $\delta-\epsilon$-definition that for given polynomial function f, $\lim_{x\to a} f(x)=f(a)$. Professor Sprague observes an advantage in expanding $f(x)-f(a)$ in powers of $x-a$, since it is $|x-a|$ which δ restricts. The method of obtaining a suitable δ which he finally presents, however, seems less simple and direct than the one illustrated, through examples, below.

Suppose one wishes to show that $\lim_{x\to 1}(x^3+5x^2)=6$. Thus, one wishes to demonstrate that for any $\epsilon>0$, a $\delta>0$ exists such that $|x^3+5x^2-6|<\epsilon$ whenever $0<|x-1|<\delta$.

Now,

$$\begin{aligned}|x^3+5x^2-6| &= |\{(x-1)^3+3x^2-3x+1\}+5x^2-6| \\ &= |(x-1)^3+8x^2-3x-5| \\ &= |(x-1)^3+\{8(x-1)^2+16x-8\}-3x-5| \\ &= |(x-1)^3+8(x-1)^2+13x-13| \\ &\leqq |x-1|^3+|8|\,|x-1|^2+|13|\,|x-1|.\end{aligned}$$

Choose δ in two stages. First, choose $\delta<1$. Then, if n is any positive integer, when $0<|x-1|<\delta$, so is $|x-1|^n<\delta$. Thus, however, when $0<|x-1|<\delta$,

$$|x-1|^3+|8|\,|x-1|^2+|13|\,|x-1| < \delta+8\delta+13\delta=22\delta.$$

Second, choose δ also so that $22\delta<\epsilon$. Apparently, if $\delta<1$ and $\delta<\epsilon/22$, then when $0<|x-1|<\delta$, $|x^3+5x^2-6|<\epsilon$.

As a second example, to show that $\lim_{x\to -2}(-3x^4-8x+2)=-30$, one has,

$$\begin{aligned}&|(-3x^4-8x+2)-(-30)| \\ &\quad = |-3(x+2)^4+24(x+2)^3-72(x+2)^2+88(x+2)| \\ &\quad \leqq |-3|\,|x+2|^4+|24|\,|x+2|^3+|-72|\,|x+2|^2+|88|\,|x+2|.\end{aligned}$$

First, choose $\delta<1$, then if $0<|x+2|<\delta$,

$$\begin{aligned}&|-3|\,|x+2|^4+|24|\,|x+2|^3+|-72|\,|x+2|^2+|88|\,|x+2| \\ &\qquad < 3\delta+24\delta+72\delta+88\delta=187\delta.\end{aligned}$$

Second, choose δ also so that $\delta<\epsilon/187$. For δ satisfying both choices,

$|(-3x^4-8x+2)-(-30)| < \epsilon$ whenever $0<|x+2|<\delta$.

For any polynomial function f, of degree at least 1, it is apparent that the method for choosing δ which is being used in the above examples, amounts only to taking $\delta<1$ and also $\delta<\epsilon/K$ where K is the sum of the absolute values of the coefficients in the expansion of $f(x)-f(a)$ in terms of $x-a$. As it turns out, students sometimes make this observation and then interest themselves in how the coefficients may be obtained by a more elegant procedure than the pedagogically rather effective—because simple—device of "insistence and adjustment" which was used in the first example. There have been classroom instances where students have invented or recalled the method for obtaining the coefficients through successive divisions of $f(x)-f(a)$ by $x-a$. Students repeating the course, to be sure, will advocate finding the coefficients via Taylor's formula.

CORRECTION

The proof of the arithmetic mean, geometric mean inequality indicated by P. H. Diananda, University of Malaya in Singapore, on page 1007, this MONTHLY, vol. 67, is to be found in D. S. Mitrinovic, *Zbornik matematickih problema*, t.I (2nd ed.), Beograd, 1958, pp. 232–233.

MATHEMATICAL EDUCATION NOTES

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ADVANCED PLACEMENT EXAMINATION OF THE COLLEGE ENTRANCE EXAMINATION BOARD

Last month Vance and Pieters reviewed the history of the development of the advanced placement examinations and pointed out their influence on new developments in mathematics and their current importance in mathematical education. Permission has been granted by the College Entrance Examination Board to publish the questions in Section I of the mathematics examination given in 1958, and in Section II of the 1959 examination.

Mathematics, Section I, 1958

Time—1 hour

Directions: Solve the following problems, using the blank pages for scratchwork. Indicate your answers on the answer sheet. No credit will be given for anything written in the examination book. Do not spend too much time on any one problem.

1. The equation of the circle with center $(2, -1)$ and radius 3 is

(A) $x^2-4x+y^2-2y=4$ (B) $x^2+4x+y^2-2y=4$ (C) $x^2-4x+y^2+2y=4$
(D) $x^2-4x+y^2+2y=14$ (E) $x^2+4x+y^2+2y=14$

2. The equation of the line whose intercepts are twice those of the line $2x-3y-6=0$ is

   (A) $2x-3y-3=0$ (B) $4x-6y-12=0$ (C) $4x-6y-6=0$
   (D) $2x-3y+12=0$ (E) $2x-3y-12=0$

3. A relative maximum value of the function $y=x^3-\frac{3}{2}x^2+2$ is

   (A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\frac{3}{2}$ (E) 2

4. If $y=2/(4+x^2)$, then $dy/dx=$

   (A) $\dfrac{2}{(4+x^2)^2}$ (B) $\dfrac{2x}{4+x^2}$ (C) $\dfrac{4x}{(4+x^2)^2}$ (D) $\dfrac{-4x}{(4+x^2)^2}$ (E) $\dfrac{-2x}{(4+x^2)^2}$

5. The distance from the point (2, 2) to the midpoint of the segment joining (2, 3) with $(-4, -1)$ is

   (A) $2\sqrt{2}$ (B) 3 (C) $\sqrt{10}$ (D) $3\sqrt{2}$ (E) $\sqrt{13}$

6. The value of k that makes the pair of lines

$$6x-9y=5$$
$$kx-4y=8$$

   perpendicular is

   (A) -6 (B) $-\dfrac{8}{3}$ (C) $\dfrac{3}{8}$ (D) $\dfrac{8}{3}$ (E) 6

7. The number of points of inflection of the curve $y=x^5+5x^3+10x+1$ is

   (A) 0 (B) 1 (C) 2 (D) 3 (E)4

8. The area between $x^2y=1$ and the x-axis from $x=\frac{1}{2}$ to $x=1$ is

   (A) $\dfrac{7}{24}$ (B) $\dfrac{1}{2}$ (C) 1 (D) $\dfrac{5}{3}$ (E) 2

9. The radius, x, of a sphere increases with the time, t. What is the radius of the sphere when the surface area ($S=4\pi x^2$) and radius are increasing at the same rate?

   (A) $\dfrac{1}{8\pi}$ (B) $\dfrac{1}{\pi}$ (C) π (D) 2π (E) 8π

10. $\int_1^e dx/x=$

    (A) 0 (B) 1 (C) e (D) $e-1$ (E) $\log_e x$

11. $\int_2^6 dx/\sqrt{x-1}=$

    (A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) 3

12. The slope of the curve $x^2+2xy+3y^2=3$ at the point $(2, -1)$ is

    (A) -2 (B) -1 (C) 0 (D) 1 (E) 2

13. An equation of the line tangent to the curve $y=\sin 2x+3\cos 2x$ at the point on the curve where $x=0$ is

    (A) $2x-y=-3$ (B) $2x-y=-1$ (C) $6x+y=1$ (D) $2x+y=3$ (E) $x-2y=-3$

14. The equation of the parabola whose focus is $(-1, 3)$ and directrix is the line $y=-1$ is

    (A) $x^2+2x=8y$ (B) $x^2+4x-y=-6$ (C) $24x^2=8y$ (D) $2x^2+5y=17$
    (E) $x^2+2x-8y+9=0$

15. A point moves along the x-axis according to the law: $x=3t^2-t^3$. When the acceleration is zero, the velocity is

(A) 0 (B) 1 (C) 2 (D) 3 (E) none of these

16. The derivative of $\log_e(x^5)$ is

(A) $5(\log_e x^6)$ (B) $5x^4(\log_e x)$ (C) $5x^5$ (D) $5x$ (E) $\frac{5}{x}$

17. If $x=2t-3$ and $y=4-5t^2$, then dy/dx is

(A) $-20t$ (B) $-10t$ (C) $-5t$ (D) $-5x$ (E) $-5(x+3)$

18. The second derivative of $f(x)=e^x+e^{-x}$ is

(A) e^x-e^{-x} (B) $2f(-x)$ (C) $f(x)$ (D) $-f(x)$ (E) $x^2f(x)$

19. The area bounded by the parabola $x=2y^2$ and the line $x=8$ is rotated around the x-axis. The volume of the solid of revolution swept out is

(A) 8π (B) 10π (C) 12π (D) 14π (E) 16π

20. $\int_0^1 xe^{x^2}dx$ is equal to

(A) e (B) e^2 (C) $\frac{1}{2}(e-1)$ (D) $2e$ (E) 1

21. $\lim_{x\to 0}(\tan 2x)/x$ is

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) ∞

22. At what point on the curve $y=\sqrt{4-x}$ is the tangent parallel to the line $x+2y=7$?

(A) $\left(\frac{63}{16}, \frac{1}{4}\right)$ (B) $(3,-1)$ (C) $(0,2)$ (D) $(3,1)$ (E) $(4,0)$

23. One of the following statements is false with regard to the graph of $y=\{x-1\}/\{x^2(x-2)\}$. Which one is false?

(A) The x-intercept is 1.
(B) There are two vertical asymptotes.
(C) There is one horizontal asymptote.
(D) There is no y-intercept.
(E) The graph is symmetric to the y-axis.

24. The minimum value of $y=2\sqrt{x-1}$ occurs at

(A) $x=-1$ (B) $x=0$ (C) $x=1$ (D) $x=2$ (E) none of these

25. The area under the curve $y=\tan x$ between $x=0$ and $x=\pi/3$ is

(A) $\log_e 2$ (B) $\cot\left(\frac{\pi}{3}\right)$ (C) $-\cot^{-1}\left(\frac{\pi}{3}\right)$ (D) $-\log\sin\left(\frac{\pi}{3}\right)$ (E) $\frac{1}{2}$

26. If $f(x)=(1-x)^3$ and $g(x)=1/x$, then the derivative of $f(g(x))$ is

(A) $3(1-x)^{-4}$ (B) $3\left(-\frac{1}{x^2}\right)^3$ (C) $3(1-x)^3$ (D) $3\left(1-\frac{1}{x}\right)^3$ (E) none of these

27. The area bounded by the curve $x=(y-1)^3$ and the straight line $x=y+1$ is

(A) $\frac{3}{2}$ (B) $\frac{9}{2}$ (C) 9 (D) $\frac{27}{2}$ (E) 27

28. The curve $y=f(x)$ between the points where $x=a$ and $x=b$ is revolved about the line $y=1$. If $f(x)>1$ for $a\leqq x\leqq b$, the volume of the solid generated is

(A) $\pi\int_a^b [f(x)]^2dx$ (B) $\pi\int_a^b ([f(x)]^2-1)dx$ (C) $\pi\int_a^b [(f(x))^2-2f(x)+1]dx$

(D) $\pi\int_a^b (f(x)+1)^2dx$ (E) none of these

29. If $y=x^2-x$, the derivative of y^2 with respect to x^2 is

(A) $2x-1$ (B) $2x^2-3x+1$ (C) $4x^3-6x^2+2x$ (D) 2 (E) $12x^2-12x+2$

30. The area in the first quadrant next to the y-axis and bounded by that axis and the curves $y=\cos x$ and $y=\sin x$ is rotated about the x-axis. The volume of the solid generated is

(A) $\frac{\pi}{2}$ (B) π (C) $\frac{\pi}{4}-\frac{1}{2}$ (D) $\frac{\pi^2}{2}$ (E) $\frac{\pi^2}{4}-\frac{\pi}{2}$

Mathematics, Section II, 1959

(In this section students are asked to show all of their work and to indicate clearly the methods used, because grades are based on the correctness of methods as well as on the accuracy of final answers. Some credit is given for partial solutions.)

Time—2 hours

1. Find the area of the region bounded by the curves $y=x-2$ and $y=2x-x^3$.
2. Consider a parallelogram and any point in its plane. Prove by analytic geometry that if the sum of the squares of the distances of the point from two opposite vertices of the parallelogram is equal to the sum of the squares of the distances of the point from the other two vertices of the parallelogram, then the figure is a rectangle.
3. The area in the first quadrant bounded above by the curve $y=\sin x$, below by the x-axis, and on the right by the line $x=\pi/2$ is divided into two equal parts by the line $x=c$. Find c.
4. A light is on top of a pole h feet high. A ball is dropped from a point at the same height as the light but k feet horizontally away from it. How fast is the shadow of the ball moving along the ground half a second later? Assume that the ball falls a distance $s=16t^2$ feet in t seconds.
5. Discuss the graph of $y=(x^2+1)e^x$. Consider
(a) the intercepts, (b) asymptototes, (c) maxima and minima, (d) points of inflection, (e) behavior for large $|x|$, (f) symmetries. Choose convenient scales and sketch the curve.
6. The graph of
$$y=\frac{ax+b}{(x-1)(x-4)}$$
has a horizontal tangent at the point $(2, -1)$. Find a and b, and show that the function has a relative maximum at this point.
7. Given a function $f(x)$ such that $f(1)=f(2)=4$, and such that $f''(x)$ exists and is positive throughout the interval $1\leqq x\leqq 3$:
(a) What can you say about the sign of $f'(3)$?
(b) Prove your statement, stating whatever theorems you use in your proof.
8. A straight fence 100 yards long stands on a ranch. The fence is to be left standing, and part or all of it is to be used in forming a rectangular corral, using an additional 260 yards of fencing for the other three sides. Find the maximum area which can be so enclosed.
9. The area bounded by the curve $y=e^{-x}$ and the lines $y=0$, $x=0$, and $x=10$ is rotated about the x-axis. Compute to three significant figures the volume of the solid of revolution so generated. Justify any approximations which you use.
10. $P(p, q)$ is a variable point on the parabola $y=x^2$. $V(0, 0)$ is the vertex of the parabola. $R(0, r)$ is the point where the perpendicular bisector of VP intersects the y-axis. Find $\lim_{p\to 0} r$.

Teacher Training Requirements in New York

On August 26, 1960, the New York State Board of Regents substantially raised the requirements for new high school teachers. The acceptable minimum of college credits in science, mathematics, and foreign language will be doubled. Dr. James E. Allen, State Commissioner of Education, is quoted in the *New York Times* of August 27 as saying that he considered this move the most dramatic and the most significant step so far in the direction of improving education. Hé said that, though the new requirements were not aimed at the often controversial "methods" courses, the inevitable effect will be to give them relatively less importance and to subordinate them to the "subject" courses. For science teachers, the science requirement under the new regulations is 42 semester hours. The new certification rules apply to teachers entering the high schools in 1963, thus giving the colleges time to make the necessary curriculum changes.

(American Institute of Physics *Educational Newsletter*, Vol. III, No. 10, August 31, 1960.)

Feasibility Study in Elementary and Junior High School Science

The work of the course-content improvement programs in biology, chemistry, geology, mathematics, and physics makes it desirable that similar attention be given to science education in the elementary and junior high schools. Study is needed on problems related to the proper placement of topics and the more specialized courses and whether fields other than the traditional ones, such as the whole area of the atmospheric and earth sciences, should be included at the elementary and junior high school levels.

Because of the widespread recognition of and interest in these problems, the American Association for the Advancement of Science has undertaken this year a preliminary study of science instruction in the elementary and junior high school years, with the support of NSF. The study was directed by a small planning group of leading scientists, teachers, and school administrators. Plans included three 2- or 3-day regional conferences, each involving approximately 40 participants. In a final report recommendations were made on the kind of studies that could contribute to the improvement of science instruction in elementary and junior high schools.

(*Science Education News*, September 1960.)

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1453. (*Corrected*) [1961, 177].

Let A be the sum of the digits of a natural number N, let $B=A+N$, let A' be the sum of the digits of the number B, and let $C=B+A'$. Find N if the digits of C are those of N in reverse order.

E 1471. *Proposed by J. L. Brenner, Stanford Research Institute, Menlo Park, California*

If A, B are invertible matrices of the same dimension, it is not always possible to solve $XY=A$, $YX=B$ for X, Y; A and B must be similar, since $X^{-1}AX=B$. Under what conditions on invertible A, B, C can one solve $XY=A$, $YZ=B$, $ZX=C$ for X, Y, Z?

E 1472. *Proposed by C. H. Cunkle, Utah State University, and W. H. Leser, Franklin and Marshall College*

In a recent text, a ring R is defined as an additive Abelian group, closed and associative under multiplication and for which the distributive laws hold, and a subset M of R is defined to be an ideal provided M is closed under addition from within ($x\in M$ and $y\in M$ implies $x+y\in M$) and closed under multiplication from within and without ($m\in M$ and $r\in R$ implies $mr\in M$ and $rm\in M$). Show that with this definition an ideal M need not be a subring of R.

E 1473. *Proposed by J. L. Pietenpol, Columbia University*

Show that there are infinitely many square triangular numbers.

E 1474. *Proposed by Alvin Hausner, City College of New York*

Show that the equation $m^{n^m}=n^{m^n}$ has no solutions in positive integers with $m\neq n$.

E 1475. *Proposed by R. H. C. Newton, Berkhamsted, England*

Let OP be a radius of a right section of a perfectly reflecting circular cylinder. Let Q be any point on OP, and let a ray leave Q and reflect from the inner surface of the cylinder. Does it ever return to Q, and if so, under what conditions?

SOLUTIONS

Las Vegas Deathbed Scene

E 1441 [1960, 1028]. *Proposed by Bart Park, Michigan College of Mining and Technology*

A resident of Las Vegas, on his deathbed, held the following conversation with a friend who liked to solve problems.

"Over the past few years I have saved every silver dollar that came into my hands. When I reached one hundred I tied them in a bag. The first three hundred accumulated pretty fast and I hoped to reach a thousand, but I didn't make it. The several bags, each containing one hundred dollars, are in the closet of the next room. What I want you to do is visit my minor son on his next and each succeeding birthday and give him the number of dollars which equals his age. I figure that on your last trip you will have just used up all of the money."

"That's interesting," said his friend. "Before I see the bags let me try to

figure out how many trips will be required." After an interval the friend said, "I'll have to know your son's age." He was told. Then he said, "I now know how many trips will be required." How many will he have to make?

Solution by G. A. Kandall, Massachusetts Institute of Technology. Let t represent the number of trips required ($t \geqq 1$) and let a represent the son's present age ($0 \leqq a < 21$). Then $ta + t(t+1)/2 \doteq 100n$, where n is an integer, $4 \leqq n \leqq 9$. It is not difficult to determine that the only possible solutions are $(t, a, n) = (25, 3, 4)$, $(25, 7, 5)$, $(25, 11, 6)$, $(25, 15, 7)$, $(35, 2, 7)$, $(25, 19, 8)$, $(40, 2, 9)$. But according to the problem, a uniquely determines t. Hence $a \neq 2$. Therefore $t = 25$.

Also solved by R. G. Albert, R. H. Anglin, J. W. Baldwin, Leon Bankoff, Merrill Barnebey, W. F. Barnett, Jeanette Bickley, D. A. Breault, R. I. Canfield, F. H. Cleveland, D. I. A. Cohen, E. C. Coolidge, Gus Di Antonio, J. W. Ellis, J. F. Foley, Robert Frandmelcrot, Michael Goldberg, L. D. Goldstone, S. H. Greene, J. E. Homer, Jr., A. R. Hyde, P. G. Kirmser, D. C. B. Marsh, D. A. Moran, S. J. Myzel, C. S. Ogilvy, Sidney Penner, Jon Petersen, J. L. Pietenpol, John Rainwater, J. R. Retherford, L. A. Ringenberg, S. W. Saunders, E. L. Spitznagel, Jr., D. C. Stevens, R. S. Strichartz, W. B. Stovall, Jr., W. C. Waterhouse, Walter Zayachkowski, and the proposer. Late solutions by Mike Brown, W. P. Cooke, Jr. and H. B. Lambert (jointly), R. B. Grafton, Glen Luchau, J. B. Muskat, C. F. Pinzka, and Guy Torchinelli.

Almost half of these solutions were incomplete or disagreed in some way with the solution published above.

The Diophantine Equation $x^2+36=y^5$

E 1442 [1960, 1028]. *Proposed by Andrzej Makowski, Warsaw, Poland*

Prove that the equation $x^2+36=y^5$ has no solution in integers x and y.

Solution by Philip Franklin, Massachusetts Institute of Technology. Suppose $x^2+36=y^5$ with x an even integer. Then y must be an even integer. Since $x=4m$ or $4m+2$, $x^2 \equiv 0$ or 4 (mod 16), and $x^2+36 \equiv 4$ or 8 (mod 16). But $y=2k$ implies $y^5 \equiv 0$ (mod 16). Thus there is no integral solution with x even.

Suppose that $x^2+36=y^5$ with x an odd integer. Then $x=2m+1$ implies $x^2 \equiv 1$ (mod 4) and $x^2+36 \equiv 1$ (mod 4). Since $y^5 \equiv 1$ (mod 4), $y \equiv 1$ (mod 4) and $y=4k+1$. Observe next that $x^2+4=y^5-2^5$ is divisible by $y-2=4k-1$. Hence, by a known theorem, the integer $n=y^5-2^5$ has no proper representation as a sum of two squares. That is, $n \neq a^2+b^2$ with $(a, b)=1$. But, since x is odd, $(x,2)=1$, and $n=x^2+2^2$ would give a proper representation. This contradiction shows that there is no integral solution with x odd.

The problem suggests some associated results and a conjecture. The results, which are elementary, are:

(a) $x^2+9=y^3$ has no integral solutions.

(b) $x^2+81=y^5$ has no integral solutions.

The argument, similar to that given, uses $x^2+1=y^3-2^3$ and $x^2+49=y^5-2^5$.

(c) $x^2+4=y^5$ has no integral solutions.

Here the residues modulo 11 show that $x^2+u \not\equiv y^5$ (mod 11) if $u \equiv 4$ (mod 11). For example $u=(11m \pm 2)^2$. Hence (b) is a special case of this.

The following conjecture is of a more general nature (and, if true, is probably much more difficult to prove):

The equation $x^2+b^2=y^5$ has no positive integral solutions with $(x, y)=1$ if $b<38$.

Here the bound is suggested by $41^2+38^2=5^5$. For this bound, the condition $(x, y)=1$ is necessary, in view of $4^2+4^2=2^5$ and $55^2+10^2=5^5$.

Also solved by R. H. Anglin, W. J. Blundon, Leonard Carlitz, Gus Di Antonio, Michael Goldberg, L. D. Goldstone, S. H. Greene, Alvin Hausner, D. C. B. Marsh, Jon Petersen, and the proposer. Late solutions by R. B. Grafton, J. B. Muskat, W. C. Waterhouse, and C. C. Yalavigi.

The proposer pointed out that by an argument analogous to that used above, one can show that the equation

$$x^2 + 4(2^{2n-1} + 1) = y^{2n+1}, \qquad n > 2,$$

has no solution in integers x and y.

A Property of the Tetrahedron

E 1443 [1960, 1028]. *Proposed by N. A. Court, University of Oklahoma*

If two of the four lines joining the vertices of a tetrahedron to the orthocenters of the opposite faces are coplanar, the remaining two lines are also coplanar.

Solution by John Rainwater, University of Washington. Let A and B be two vertices, H_a and H_b the opposite orthocenters, P and Q the other two vertices, A' and B' the projections of A and B on PQ. Then A, H_b, A', and also B, H_a, B' are collinear. We *assume* (which the proposer failed to do) that $A \neq H_b$, $B \neq H_a$. Then: (AH_a, BH_b coplanar)$\leftrightarrow$(A, B, H_a, H_b coplanar)$\leftrightarrow$(AH_b, BH_a coplanar) $\leftrightarrow$(AA', BB' coplanar)$\leftrightarrow$($A'=B'$)$\leftrightarrow$(AB in a plane, namely ABA', perpendicular to PQ)$\leftrightarrow$(AB perpendicular to PQ). This last condition is symmetric in the pairs (A, B) and (P, Q).

Examples are easily constructed showing that the statement may be false when $A=H_b$ or $B=H_a$.

Also solved by V. F. Ivanoff, D. C. B. Marsh and the proposer.

Editorial Note. One must not confuse the lines joining the vertices of a tetrahedron to the orthocenters of the opposite faces with the altitudes of the tetrahedron. It is known that if two altitudes of a tetrahedron are coplanar, then the other two altitudes are also coplanar (see Art. 205 of Court's *Modern Pure Solid Geometry*, (1935)).

A Matrix Identity

E 1444 [1960, 1028]. *Proposed by Lincoln Teng, Willow Run Laboratories, University of Michigan*

Let A and B, two $n\times n$ matrices, be such that A and $AB-BA$ commute. Show that $n(A^kB-BA^k)=k(A^nB-BA^n)A^{k-n}$ for any integer $k>n$.

Solution by Leonard Carlitz, Duke University. We have first, for any positive integer k,

$$A^kB - BA^k = k(AB - BA)A^{k-1}. \tag{1}$$

Indeed, (1) is obvious for $k=1$, and assuming it holds for a particular k, then

$$\begin{aligned} A^{k+1}B - BA^{k+1} &= A(A^kB - BA^k) + (AB - BA)A^k \\ &= kA(AB - BA)A^{k-1} + (AB - BA)A^k \\ &= (k+1)(AB - BA)A^k. \end{aligned}$$

Next, if $k>n>0$, then, by (1),

$$k(A^nB - BA^n)A^{k-n} = kn(AB - BA)A^{k-1} = n(A^kB - BA^k),$$

regardless of whether n is the order of the matrices.

Also solved by R. G. Albert, D. A. Breault, G. D. Chakerian, R. W. Cottle, L. D. Goldstone, S. H. Greene, Jiang Luh, D. C. B. Marsh, F. D. Parker, W. V. Parker, Jon Petersen, J. L. Pietenpol, Thomas Porsching, D. A. Robinson, D. W. Robinson, and W. C. Waterhouse. Late solutions by T. H. Slook, D. H. Skypek, and the proposer.

Relation (1) is known and appears, e.g., on p. 61 of P. R. Halmos, *Finite-Dimensional Vector Spaces* (2nd ed.). As was pointed out by D. A. Robinson, the same proof as given above establishes the general

THEOREM *If A and B are elements in a ring $\{R, +, \cdot\}$ such that A and $AB-BA$ commute and if k and n are positive integers such that $k \geqq n$, then $n(A^kB-BA^k)=k(A^nB-BA^n)A^{k-n}$.*

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4971. *Proposed by D. J. Newman, Yeshiva University*

Does the infinite matrix take l^2 vectors into l^2 vectors?

$$\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{3} & \cdots \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{vmatrix}$$

4972. *Proposed by P. T. Bateman, University of Illinois*

If $P(x)$ is a polynomial with real coefficients but no zeros in (0, 1), it is known that the power series for $(1-x)^{-r-1}P(x)$ has positive coefficients for all

sufficiently large positive integers r. How large must r be taken if $P(x)=1-20x+101x^2$?

4973. *Proposed by Albert Wilansky, Lehigh University*

The sequence $\{\cos nx\}$ is topologically free on $[0, \pi]$, *i.e.*, no one of its members is uniformly approximable by a linear combination of the others. Is the same true on $[0, a]$ if $0<a<\pi$?

4974. *Proposed by P. M. Cohn, The University, Manchester, England*

Does there exist a field (commutative or not) whose multiplicative group is a free group?

4975. *Proposed by Robert Spira, Berkeley, California*

Prove or disprove: If $\mathrm{Re}(z)\geqq 1$, then for any positive integer n,

$$|z^{n+1}-1| > |z^n|\cdot|z-1|.$$

The result has been checked for $n<10$.

4976. *Proposed by H. A. Pogorzelski, American Mathematical Society, Providence, R. I.*

We employ the notation $2\uparrow n$ in lieu of 2^n and represent the expressions $2\uparrow 2$, $2\uparrow(2\uparrow 2)$, $2\uparrow(2\uparrow(2\uparrow 2))$, $\cdots$ by $2\downarrow 2$, $2\downarrow 3$, $2\downarrow 4$, $\cdots$, respectively; that is, $2\downarrow n$ is defined recursively by

$$2\downarrow 0 = 1, \qquad 2\downarrow(n+1) = 2\uparrow(2\downarrow n).$$

(This operation is called tetration by R. L. Goodstein.) The associative law does not hold in general for $2\uparrow 2\uparrow 2\uparrow\cdots\uparrow 2$. Prove the following conjecture: If n is an integer $\geqq 3$, then the various ways of associating within the exponent-chain $2\uparrow 2\uparrow\cdots\uparrow 2$ of n 2's give a set of $2\uparrow(n-3)$ different integral values of which the minimum is $2\uparrow(2\uparrow(n-1))$ and the maximum is $2\downarrow n$.

SOLUTIONS

Sections of a Cylinder of Revolution

3610 [1944, 243]. *Proposed by W. H. Rasche.*

Determine the position of a cylinder of revolution which cuts the vertical and horizontal planes in equal ellipses similar to a given ellipse.

Solution by C. S. Ogilvy, Hamilton College, Clinton, N. Y. If the problem is taken to mean all three coordinate planes, we note that there is in general no solution; for only one line (essentially) makes equal angles with all three, and the shape of the ellipse is not variable. Therefore we choose the two planes XY and ZY without loss of generality.

If the given ellipse has eccentricity e, then the solution is that the axis of the cylinder must have direction components $(1, k, 1)$ (to within sign,) where $k^2=(2e^2-1)/(1-e^2)$. One way to achieve this is to choose the point

$P(\sqrt{(1-e^2)}, \sqrt{(2e^2-1)}, \sqrt{(1-e^2)})$. Then $|OP| = 1$, and the lengths of the projections of OP on both the XZ and the YZ planes are each e. Thus the acute angle which OP makes with each of these two planes is $\cos^{-1} e$. But, as proved in problem E-1353 [1959, 726], this is exactly the required angle.

It is interesting to observe that real k requires $e \geqq 2^{-1/2}$, where of course equality means that the axis of the cylinder meets both planes at 45°. Any less eccentric ellipse in one plane cannot be matched in the other.

A Diophantine Equation

4666 [1955, 734; 1957, 54; 1960. 1034]. *Proposed by R. Venkatachalam Iyer, Trivandrum, India*

If $T_p = p(p+1)/2$, solve in integers the equation

$$\frac{1}{T_x} + \frac{1}{T_y} = \frac{1}{T_z}.$$

Comment by J. J. Schäffer, Institute of Mathematics and Statistics, Montevideo, Uruguay. The alleged proof [1960, 1034] contains an unfortunate oversight which renders it invalid. After reducing the problem to the Diophantine equation

$$(ab - c^2)^2 = c^4 + \{(a-b)^2 - 2\}c^2 + 1,$$

the solver states that the right member is a quadratic expression in c^2, and so will be a perfect square, if *and only if* its discriminant $\{(a-b)^2-2\}^2-4$ vanishes, whence $a-b = \pm 2$ or 0.

The necessity of the condition, however, does not follow, since what is required is a perfect square number rather than the square of a polynomial in c^2. Furthermore, the conclusion is false, as taking $c=3$, $a-b=5$ will illustrate (also $c=4$, $a-b=5$ or 28, etc.)

These illustrations are not, unfortunately, counterexamples to the purported lemma that $a-b$ must be 0 or ± 2. The original problem, therefore, remains only partially solved without proof of the lemma or effective counterexample.

Subset of Measure Zero of the Unit Interval

4901 [1960, 382]. *Proposed by Albert Wilansky, Lehigh University*

Let the class of subsets of measure zero of the interval [0, 1] be partially ordered by inclusion and let M be a maximal chain. Is the cardinal number of M greater than that of the continuum? Is the union of the sets in M measurable?

Solution by S. Khabbaz and G. Stengle, Lehigh University. Let A be an arbitrary subset of [0, 1]. Then A can be initially ordered *i.e.* well ordered in such a way that every segment is countable. Thus A can be expressed as the union of a chain of countable sets, hence sets of measure 0. Choosing in particular A to be nonmeasurable, the first question is answered in the negative.

Let A be a set of measure zero having the cardinality c of the continuum.

Proposition c_{84} of Sierpinski, *Hypothèses du Continu* shows that A contains the union of a chain containing more than c subsets. Granting the continuum hypothesis, such a chain contains 2^c sets.

The authors are indebted to J. A. Schatz for suggesting the reference to Sierpinski.

Trinomials in $FG(2)$

4915 [1960, 597]. *Proposed by S. W. Golomb, California Institute of Technology*

Show that if the trinomial $f(x)=x^n+x^a+1$ over $GF(2)$, with $0<a<n$, has any repeated factors, then $f(x)$ is a perfect square.

I. *Solution by Joe Lipman, Harvard University.* Denote the highest (lowest) even (odd) power term in a polynomial by *he, ho, le, lo,* respectively. Put $f(x)=P_1(x)\cdot P_2(x)$ with $P_1(x)$ a square, *i.e.*, a sum of even powers. Any factor of $f(x)$ has at least two terms, one of which is 1. Consequently $P_1(x)\cdot P_2(x)$ has at least four terms, viz., $he_1\cdot he_2$, $he_1\cdot ho_2$, $le_1\cdot le_2$, $le_1\cdot lo_2$, unless $P_2(x)$ has no odd power terms, *i.e.*, unless $P_2(x)$ is also a perfect square. The conclusion follows.

II. *Solution by C. H. Franke, Rutgers—The State University, Newark, N. J.* It is sufficient to show that $(f,f')\neq 1$ implies n, a both even. Then if f has a repeated factor, $(f,f')\neq 1$ and $f(x)=(x^{n/2}+x^{a/2}+1)^2$.

If n, a both odd, $f'(x)=x^{n-1}+x^{a-1}$ and $f(x)+x\cdot f'(x)=1$ so that $(f,f')=1$. If n is odd and a even, $f'(x)=x^{n-1}$ so that $(f,f')=1$ since $f(0)\neq 0$: and similarly for n even, a odd.

Generalization is easy.

Also solved by W. J. Blundon, L. Carlitz, R. A. Cuninghame-Green, J. B. Kelly, A. G. Konheim, D. C. B. Marsh, H. F. Mattson, K. A. Post, R. F. Rinehart, and the proposer.

Relations Implying that two Matrices Be Normal

4916 [1960, 597]. *Proposed by Olga Taussky, California Institute of Technology*

Fuglede's theorem (*Proc. Nat. Acad. Sci.*, 36 (1950), 35–40) specialized to finite matrices over the complex numbers states: for A normal the relation $AB=BA$ implies $A^*B=BA^*$. Putnam's generalization (*Amer. Journ. Math.*, 73 (1951), 357–362) states: if A_1, A_2 are normal and if $A_1B=BA_2$, then $A_1^*B=BA_2^*$.

Find, conversely, a necessary and sufficient condition on a nonsingular n by n matrix B such that the relations $A_1B=BA_2$ and $A_1^*B=BA_2^*$ imply that A_1 and A_2 are normal.

Solution by the proposer. The condition is that B^*B does not have any multiple characteristic root. For, let this condition hold. Then multiply $A_1B=BA_2$ on the left by B^*. This implies

$$B^*A_1B = B^*BA_2 = A_2B^*B.$$

Let U be a unitary matrix which transforms B^*B into diagonal form. Since the elements in this diagonal matrix are different, A_2 must transform to diagonal form simultaneously. This implies that A_2 is normal. Similarly, it follows that A_1 is normal.

To show that the condition is necessary take a nonsingular matrix B which has at least one multiple characteristic root. A not-normal matrix A_2 can be determined such that $B^*BA_2=A_2B^*B$. Then determine A_1 such that $A_2B^*=B^*A_1$. This implies $A_2B^*B=B^*A_1B=B^*BA_2$. Hence $A_1^*B=BA_2^*$ and $A_1B=BA_2$.

Two Definite Integrals

4917 [1960, 699]. *Proposed by L. Lewin, Enfield, Middlesex, England*

Prove

$$\int_0^{\pi/6} \log^2(2\sin x)\,dx = 7\pi^3/216. \tag{1}$$

$$\int_0^{\pi/6} x\log^2(2\sin x)\,dx = 17\pi^4/25920. \tag{2}$$

Solution by W. H. J. Fuchs, Cornell University. Let C be the closed curve obtained by stringing together

$$C_1\colon\ z = x \qquad 0 \leqq x \leqq 1;$$

$$C_2\colon\ z = e^{i\theta} \qquad 0 \leqq \theta \leqq \pi/6;$$

$$C_3\colon\ z = e^{it/2}(2\cos t)^{1/2} \qquad \pi/3 \leqq t \leqq \pi/2,$$

with a small indentation at $z=1$ to exclude $z=1$ from the interior of C. By Cauchy's theorem,

$$\begin{aligned} 0 &= \int_C \log^2(1-z^2)\,dz/z \\ &= \int_0^1 \log^2(1-x^2)\,dx/x + i\int_0^{\pi/6} \{\log|2\sin\theta| + i(\theta-\pi/2)\}^2 d\theta \\ &\quad + \int_{\pi/3}^{\pi/2} \{\log(-e^{2it})\}^2\left(\frac{i}{2} - \frac{1}{2}\tan t\right) dt. \end{aligned}$$

Taking imaginary parts:

$$0 = \int_0^{\pi/6} \log^2(2\sin\theta)\,d\theta - \int_0^{\pi/6} (\theta - \tfrac{1}{2}\pi)^2\,d\theta - \frac{1}{2}\int_{\pi/3}^{\pi/2} (2t-\pi)^2\,dt;$$

$$\int_0^{\pi/6} \log^2(2\sin x)\,dx = 7\pi^3/216.$$

In the same way, equating to 0 the real part of $\int_C \log^3(1-z^2)dz/z$:

$$0 = \int_0^1 \log^3(1-x^2)dx/x + \int_0^{\pi/6}\left\{-3\log^2(2\sin\theta)\cdot\left(\theta-\frac{\pi}{2}\right)+\left(\theta-\frac{\pi}{2}\right)^3\right\}d\theta$$

$$+\frac{1}{2}\int_{\pi/3}^{\pi/2}(2t-\pi)^3dt.$$

$$\int_0^1 \log^3(1-x^2)dx/x = \frac{1}{2}\int_0^1 (1-y)^{-1}\log^3 y\cdot dy = \frac{1}{2}\sum_{n=0}^{\infty}\int_0^1 y^n\log^3 y\cdot dy$$

$$= -\frac{1}{2}\sum_{k=1}^{\infty} 6k^{-4} = -\pi^4/30.$$

Since now all other terms are known, we find

$$\int_0^{\pi/6}\theta\log^2(2\sin\theta)d\theta = 17\pi^4/25920.$$

Note by P. J. de Doelder, Technical University, Eindhoven, Netherlands. The proposer's solution can be found in his *Dilogarithms and Associated Functions*, McDonald, London (1958). The first of the proposed integrals equals $-\frac{1}{2}Ls_3(\pi/3)$ by (6.57), and by a formula on p. 152: $Ls_3(\pi/3) = -7\pi^3/108$, which gives (1). Formula (7.72), p. 185, gives

$$Ls_4^{(1)}(\pi/3) = -\int_0^{\pi/3} y\log^2(2\sin\tfrac{1}{2}y)dy = -17\pi^4/6480,$$

from which (2) follows upon substitution of $y=2x$.

Ab-integration in a Ring

4920 [1960, 700]. *Proposed by Smbat Abian, University of Pennsylvania*

An *Ab*-integration in a ring A is a mapping $a\to a^*$ of A into itself satisfying the following conditions:

$$(a+b)^* = a^* + b^*, \qquad (a^*b + ab^*)^* = a^*b^*, \qquad \text{for any pair } a, b \in A.$$

Is there a nontrivial *Ab*-integration in the field of real numbers?

Solution by D. C. Stevens, Cornell University. There is. Set $1^*=s$, and $(1/1^*)^*=t$, where s and t are algebraically disjoint over the rationals.

From the given first condition, it is easy to verify that $(rx)^*=rx^*$ for r a rational. Further, $(1\cdot 1^*+1^*\cdot 1)^*$ gives $2s^*=s^2$ and by induction, $(s^n)^* = s^{n+1}/(n+1)$ for integral $n\geqq 0$; also $(t/s)^*=t^2/2$.

Hence $s\leftrightarrow x$ (the usual variable in analysis), $t\leftrightarrow\log x$, and * corresponds to integration, is an isomorphism between certain functions and certain reals. For reals y which are not polynomials in s and t we may set $y^*=0$.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of The Association.

An Introduction to the Theory of Numbers. By Ivan Niven and H. S. Zuckerman. Wiley, New York, 1960. viii+250 pp. $6.25.

This is a well-written, carefully organized text for a first course in number theory at the senior or graduate level. There are chapters on divisibility, congruences, quadratic reciprocity, number-theoretic functions, diophantine equations, Farey fractions, continued fractions, distribution of primes, algebraic numbers, partitions and density of sequences. The content of the final three chateers is not ordinarily to be found in a book at this level. There are in fact many unusual and praiseworthy sections throughout the book, such as those conneciing the notions of elementary congruence theory and of modern algebra, that giving a proof of Legendre's necessary and sufficient conditions that $ax^2+by^2+cz^2=0$ have a nontrivial solution, and those on numerical computations. The collection of problems is an outstanding feature; there are over 500, carefully selected to illustrate the text or to lead the student in new directions. In summary, this should remain one of the standard texts in the subject for many years.

W. J. LEVEQUE
University of Michigan

Modern Probability Theory and its Applications. By Emanuel Parzen. Wiley, New York, 1960. xv+464 pp. College $9.00, Trade $10.75.

The book is designed primarily as a text for a basic probability course of a semester's duration, on the junior-senior level. A somewhat shorter course may be taught out of the book for students with only a year of calculus as background. On the other extreme, there are two chapters at the end of the book which are of a more sophisticated nature than normally taught in undergraduate courses.

The book starts with fundamental definitions, and works its way—at a rather leisurely pace—through discrete spaces, to random variables in one and n dimensions. It treats all the usual topics in a probability course, ending with a careful treatment of sums of independent random variables and limit theorems for sequences of random variables. The author places great emphasis on clarity of presentation, perhaps too much emphasis.

The book's greatest asset is the excellent collection of illustrative examples and varied exercises. They are selected from a wide variety of fields of interest to the pure mathematician or to scientists. While many of these are well known, there are also a pleasing number of new applications of probability theory.

The author has also been wise in his choice of when to prove theorems, and which theorems to state without proof. In many cases theorems are stated and proved in a form sufficiently strong for most applications and yet sufficiently far from the most general known theorem to allow elementary treatment.

The book's weakest part is the first chapter. It has become customary in probability books to leave a few basic terms undefined. While the author is to be commended for attempting to fill this gap, his efforts at defining such terms as "random phenomenon" and "probability" are not very successful. For example, according to Parzen's definition a sequence that alternates is a random phenomenon. However, if we observe it only every second time, it is not a random phenomenon. This leads to great difficulties.

Equally troublesome, from the point of view of applications, is the fact that one does not know whether an event is a random event until one has observed it infinitely often (since it is random only if the frequency of occurrence tends to a limit). Thus we can never know in practice whether probability theory is applicable to a sequence of experiments. If one reads these definitions literally, and if the universe should happen to have a finite future, then probability theory is in principle inapplicable. However, the value of the book is not destroyed if these early definitions are ignored.

The book may prove itself valuable as a college text, especially for prospective scientists and engineers. It will certainly be a very valuable reference work for applied mathematicians.

JOHN G. KEMENY
Dartmouth College

The Foundations of Mathematics. By Evert W. Beth. North-Holland, Amsterdam, 1959. xxvi+741 pp. $13.25.

Work in the foundations of mathematics has passed out of the realm of pure philosophy and logic into the domain of the mathematician. This is not to say that philosophical issues are unimportant, but rather to underline the increasing importance of the work being done in foundations for the mathematician as such. Indeed, as Professor Beth points out, there is a tendency for certain areas of work in foundations to be considered apart from their relevance to the broad problem of a precise and well-formulated foundation for mathematics. I need but mention two areas of importance to illustrate what I mean—axiomatic set theory and recursive functions.

The increasing amount of research in this interesting and important area calls for a book that may serve as a comprehensive survey of the field. The present book provides precisely what is needed, for it gives an over-all view of the development of foundation theory as well as brief accounts of recent results. The book begins with a short historical section, after which the number system is constructed on the basis of the form of a deductive system. Metamathematical questions lead to the analysis of the concept of existence in mathematics. Set theory, its applications to completeness theorems and models, is considered after

a survey of the paradoxes. The decision problem and the related theory of recursive functions are extensively developed. The final section is devoted to philosophical considerations expressing Professor Beth's views. The bibliography is quite good and should serve as an excellent guide to students seeking additional sources. An Index of Authors, an Index of Subjects, and a detailed Table of Contents make the material in the book readily accessible.

LOUIS O. KATTSOFF
Boston College

Classical Dynamics. By R. H. Atkin. Wiley, New York, 1959. ix+273 pp. $5.25.

This book has as its avowed aims those of teaching the student dynamics and to answer examination questions—English examination questions, that is. The reviewer, like the author, does not believe these aims to be mutually exclusive, but unlike the author (apparently), does not believe this textbook suitable to their attainment. The book is traditional nineteenth century dynamics at its most pedestrian level. Relieving features are the use of vector notation and occasional recourse to matrices. It largely neglects the physical aspects of dynamics, but does not have the saving grace of setting forth with any notable clarity or precision the mathematical structure of the subject. It cannot be said, however, that problem solving is neglected. On the contrary, one gets the impression that the treatise consists of solved examples with a thin, perfunctory connective tissue of discussion of general principles. For example: there are four central chapters labelled *Kinetics of particle motion* which consist of 43 pages of worked-out examples and 15 pages of theory. (In fairness, it should be said that the examples frequently do embody portions of what may be called theory.) Every chapter is concluded with a large number of exercises taken from Tripos papers and Love's *Theoretical Mechanics.*

The reviewer feels that this approach to dynamics is conducive neither to real understanding nor to the excitation of an interest in this discipline. Indeed, the bright mathematics student may be so repelled as to develop a lasting aversion for all applications of mathematics and so to exacerbate the all too existent applied-pure mathematics dichotomy present in Western Europe and America.

WILLIAM H. PELL
National Bureau of Standards

Pure Mathematics. By F. Gerrish. Vol. I, *Calculus.* xxv+1 to 361+23 pp. Vol. II, *Algebra, Trigonometry, Coordinate Geometry.* xxviii+363 to 758+25 to 48 pp. Cambridge University Press, New York, 1960. Vol. I, $5.00, Vol. II, $6.50.

These are British books intended to prepare the student for a certain degree at the University of London without being "cram" books. It would be out of place for an American reviewer writing for American readers to attempt to

judge how well they fulfill their purpose. Rather it is appropriate to describe the books and to comment on their usefulness in American situations.

They are not "cram" books. They are sophisticated and rigorous. Their conciseness and extensive coverage remind one of textbooks like Fine's *College Algebra*, long out of style in America. Their content is of the traditional type. Set theory and Modern Mathematics with capital M's are definitely missing.

One can hardly imagine these books being used as texts in America, even in courses with the same titles as the books. In this country some of the content of Volume II would be placed before Volume I, some would be placed after and some would not be used at all. Volume I covers approximately the content of a traditional American first course in the calculus including partial derivatives and a long chapter on differential equations, but excluding series and multiple integrals. Volume II contains much equation theory including complex numbers, determinants, and Cramer's rule, and a long, thorough chapter on series. Also in this volume a long course in coordinate geometry with much work on conics precedes the final chapter on spherical trigonometry and follows one on De-Moivre's theorem applied to roots of equations and to series with complex terms.

A bright student could use these books as sources of enrichment or a teacher could use them for reference. For instance, Volume I contains such materials as the proof that the arithmetic mean is always equal to or greater than the geometric mean, page 8, Euler's constant, page 123, the use of the binomial theorem to show that, as n increases without bound, $(1+1/n)^n$ has a limit between $2\frac{1}{4}$ and 3, page 57, the distinction between derivation and differentiation, page 89. Volume II contains such goodies as a neat proof of Ceva's and Menelaus's theorems by coordinate geometry, page 574, length of common perpendicular between two skew lines, page 728, many relations between circular and hyperbolic functions, pages 557–60. The problem of the circle orthogonal to three circles in all its cases is efficiently disposed of in one half of page 592.

These books could also be used for review by students already somewhat familiar with the material. The author claims that they are suitable for students working alone. The learning helps he gives are (a) an introductory review chapter in each volume, (b) helpful warnings of errors easy to make, (c) expositions of fine distinctions not mentioned in many texts and (d) conciseness and clarity in writing. Missing are the long heuristic discussions which help build concepts in modern American texts.

Mechanical features of the books worth mentioning are (a) use of different kinds of type and starred material for greater flexibility, (b) use of lower case delta instead of upper case delta in designating increments, (c) consecutive numbering of pages through the two volumes as if they were one, (d) designation of answers to odd-numbered problems at the backs of the books by page number as well as by section number, (e) long, useful indexes.

ANICE SEYBOLD
North Central College

Handbook of Laplace Transformation. By Floyd E. Nixon. Prentice-Hall, Englewood Cliffs, N. J., 1960. ix+115 pp. $4.50.

Nearly three-sevenths of the book is devoted to " . . . the most modern and extensive table of transform pairs available today, all number-coded for quick and easy reference." These give the inverse transforms of proper rational fractions. Besides derivations of relationships and illustrations of applications to ordinary, linear, differential equations the mathematics usually encountered is clearly illustrated. Although not advanced, the book should prove to be helpful.

Formula (3.10–13) should have the summation sign in the denominator. Assuming that $f(t-a)$ on pages 30 and 65 is automatically zero for $t<a$ is unconventional, as is the use of $\int_{0+}^{t} f(t)dt$ for $f^{(-1)}(t)$ on pages 52 and 65.

EARL LAFON
University of Oklahoma

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor J. D. Mancill, University of Alabama, represented the Association at the inauguration of Dean Wallace Colvard as President of Mississippi State University on April 15, 1961.

Professor G. M. Merriman, University of Cincinnati, represented the Association at the inauguration of Dr. J. M. Read as President of Wilmington College on April 30, 1961.

Professor C. T. Salkind, Polytechnic Institute of Brooklyn, represented the Association at the inauguration of Dr. J. R. Everett as Chancellor of the Municipal College System of the City of New York on April 24, 1961.

Emory University: Assistant Professor E. E. Grace will be on leave for the summer of 1961 and the academic year 1961–62 at the University of Wisconsin as a National Science Foundation Science Faculty Fellow; Professor Tomlinson Fort, University of Miami, has been appointed Professor; Dr. Mary Neff has been appointed Assistant Professor; Mr. Marshall Saade has been appointed Instructor.

Mr. A. A. Benvenuto, System Development Corporation, Santa Monica, California, has accepted the position of Operations Research Analyst at the System Development Corporation, Falls Church, Virginia.

Mr. F. L. Coling, Raytheon Manufacturing Company, Bedford, Massachusetts, has accepted a position as Engineer with the Douglas Aircraft Company, Missile Division, Santa Monica, California.

Mr. W. P. Durbin has accepted a position as Senior Aerosystems Engineer with Convair, Fort Worth, Texas.

Dr. T. C. Fry, Remington Rand, New York, New York, has accepted a position with the National Center for Atmospheric Research, Boulder, Colorado.

Mr. Peter Lees, Onelunga High School, Auckland, SES, New Zealand, has accepted a position as Quality Control Officer with the Waitomo Portland Cement Company, Te Kuiti, New Zealand.

Mr. Joaquin Loustaunau, Oklahoma State University, has accepted a position at the Instituto Tecnologico y de Estudios Superiores de Monterrey, Monterrey, Mexico.

Mr. J. P. Menard, National Bureau of Standards, Washington, D. C., has accepted a position as Assistant Director of the Computing Center at Syracuse University.

Mr. A. M. Peiser, M. W. Kellogg Company, New York, New York, has been appointed Staff Consultant in Mathematics.

Associate Professor Alice T. Schafer, Connecticut College, has been promoted to Professor.

Mr. C. J. Smith, Remington Rand UNIVAC, St. Louis, Missouri, has accepted a position as Associate Engineer in the Scientific Data Processing Department of the McDonell Automation Center, St. Louis, Missouri.

Miss Eugenia I. Trapp, International Business Machines, Los Angeles, California, has been appointed Staff Instructor at International Business Machines, New York, New New York.

Mr. Byron Cosby, Sr., Educational Service Bureau, Columbia, Missouri, died February 3, 1961. He was a Charter Member of the Association.

Mr. P. L. Poston, Great Lakes Mutual Life Insurance Company, died August 22, 1960. He was a member of the Association for eleven years.

Professor Emeritus W. D. Reeve, Columbia University, died February 16, 1961. He was a Charter Member of the Association.

Professor Emeritus S. E. Urner, Los Angeles State College, died January 30, 1961. He was a Charter Member of the Association.

PAU SCIENCE DIVISION SPONSORS OPERATION CLEAN-OUT-THE-ATTIC

Under a new program organized by the Division of Science Development of the Pan American Union, General Secretariat of the Organization of American States, thousands of issues of scientific and technical journals are being transferred from the overcrowded libraries of United States scientists to the understocked shelves of the libraries and information centers of Latin America. The rapid development of the sciences in that area make availability of journal files essential. Scientists of the U.S.A. are being asked to donate to the U.S.B.E. files of scientific and technical journals which they no longer wish to keep in their libraries. The value of the gift plus the cost of making it (mailing charges in this case) are income-tax deductible. Further information about the program may be obtained from the U. S. Book Exchange, 3335 V. St., N.E., Washington 18, D. C., or from the Division of Science Development, Pan American Union, Washington 6, D. C.

GRADUATE TRAINEESHIPS IN BIOMETRY

Training programs designed to prepare students in the application of statistical and mathematical methods to biological problems, particularly those related to health and medical sciences, now exist in more than 20 universities throughout the country. Supported by training grant funds from the Public Health Service, NIH, these programs provide unusual opportunities for careers in teaching, research, and consultation. Employment opportunities for biometricians are excellent, with the demand by governmental and voluntary health agencies, medical research and educational institutions, and industry running far in excess of the available supply of trained personnel.

Programs of study are individually designed to lead to doctoral degrees, and in special instances, to other academic degrees. Traineeship stipends are provided at various levels depending on previous education and experience of the trainee and include allowances for

dependents. Substantially full economic support or partial support may be provided, depending upon the proportion of time spent in training.

Interested applicants are encouraged to correspond with one or more of the Program Directors listed below because course offerings, as well as specific research problems for application of learned skills, vary from school to school.

Dr. Virgil Anderson
Purdue University
Lafayette, Indiana

Dr. George F. Badger
Western Reserve University
Cleveland 6, Ohio

Dr. Jacob E. Bearman
University of Minnesota
Minneapolis 14, Minnesota

Dr. Antonio Ciocco
University of Pittsburgh
Pittsburgh 13, Pennsylvania

Dr. Wilfrid J. Dixon
Medical Center, U.C.L.A.
Los Angeles 24, California

Dr. W. T. Federer
Cornell University
Ithaca, New York

Dr. John W. Fertig
Columbia University
New York 32, New York

Dr. Franklin A. Graybill
Colorado State University
Fort Collins, Colorado

Dr. Bernard G. Greenberg
University of North Carolina
Chapel Hill, North Carolina

Drs. John Gurland & T. A. Bancroft
Iowa State University
Ames, Iowa

Dr. Boyd Harshbarger
Virginia Polytechnic Institute
Blackburg, Virginia

Dr. Allyn Kimball
Johns Hopkins University
Baltimore 5, Maryland

Dr. Schuyler G. Kohl
State Univ. of N.Y., Col. of Med.
Brooklyn 3, New York

Dr. Robert F. Lewis
Tulane University
New Orleans 12, Louisiana

Dr. Eugene Lukacs
The Catholic University of America
Washington, D. C.

Dr. Paul Meier
University of Chicago
Chicago 37, Illinois

Prof. Felix Moore
University of Michigan
Ann Arbor, Michigan

Dr. Lincoln E. Moses
Stanford Medical School
Stanford, California

Dr. Hugo Muench
Harvard School of Public Health
Boston 15, Massachusetts

Dr. Robert Quinn
Vanderbilt University
Nashville 5, Tennessee

Prof. J. A. Rigney
North Carolina State College
Raleigh, North Carolina

Drs. W. W. Schottstaedt and James Hagans
University of Oklahoma
Oklahoma City 4, Oklahoma

Dr. Malcolm E. Turner, Jr.
Medical College of Virginia
Richmond, Virginia

Dr. Colin White
Yale School of Medicine
New Haven, Connecticut

Dr. J. Yerushalmy
University of California
Berkeley 4, California

For those unable to train during the academic year, an unusual opportunity is provided by a cooperative Graduate Session of Statistics in the Health Sciences sponsored by these Program Directors and made possible by a training grant from the PHS, NIH. For information concerning available stipends and course offerings at elementary, intermediate, or advanced levels for the summers of 1961 and 1962, write Dr. Jacob E. Bearman, University of Minnesota, Minneapolis, Minnesota.

REPORT OF THE SECOND CONFERENCE ON MATHEMATICAL EDUCATION IN SOUTH ASIA

The report of this conference held at Bombay in January 1960 contains invited addresses by a number of distinguished mathematicians (Artin, Stone, Krull, Lichnerowicz, Moise, Newman, Alexandrov, and Gnedenko) together with brief accounts of the discussion at the meetings of the working groups.

Copies of the report (xxiii+205 pages, paper bound) may be purchased for $2.00 by sending remittances to the Mathematical Association of America, University of Buffalo, Buffalo 14, New York. The price quoted is a special rate for orders placed through the office of the Association and does not apply to orders placed through other agencies.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE PROPOSED DOCTOR OF ARTS DEGREE

At their meetings in Washington, D. C. on January 25, 1961 the Board of Governors of the Association and the Council of the American Mathematical Society both voted to approve, in principle, a report which recommended the establishment of a new graduate degree in mathematics. The proposal was described to both of these bodies at their meetings in East Lansing in August 1960, and a joint committee was asked to study the matter and report in January. The committee consisted of E. E. Moise, Chairman, M. M. Day, P. R. Halmos, and A. D. Wallace.

While the name of the degree was not considered part of the substance of the Committee's proposal, the Board of Governors approved a motion that the preferred name for the new degree be the Doctor of Arts.

A detailed report on the Committee's recommendations appears in the April, 1961, *Notices of the American Mathematical Society*. Reprints of this article and further information can be obtained by writing to Professor R. J. Wisner, Executive Director, Committee on the Undergraduate Program in Mathematics, Michigan State University Oakland, Rochester, Michigan.

HENRY L. ALDER, *Secretary*

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirty-eight annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the Buena Vista Hotel, Biloxi, Mississippi, on February 17–18, 1961, with Mississippi State University as host institution. The Friday afternoon meeting was held in two concurrent sessions. Professor J. H. Wahab, Louisiana Vice-Chairman, and Professor Elinor Walters, Mississippi Vice-Chairman, presided. Professor Z. L. Loflin, Chairman of the Section, presided at the Friday evening

and Saturday morning sessions. There were 206 persons registered, including 97 members of the Association.

The following officers were elected for the coming year: Chairman, Professor B. O. Van Hook, Mississippi Southern College; Vice-Chairman for Louisiana, Professor W. B. Temple, Louisiana Polytechnic Institute; Vice-Chairman for Mississippi, Professor S. R. Knox, Millsaps College; Secretary-Treasurer, Professor Z. L. Loflin, University of Southwestern Louisiana.

At the business meeting a report on the progress of the High School Contests was given by Professor N. A. Childress, Chairman of the Committee on Contests and Professor H. T. Karnes, Chairman for Louisiana. Professor Arthur Ollivier reported on the meeting of the Board of Governors.

The invited speaker for the meeting was Professor C. W. Curtis of the University of Wisconsin. His lecture on Friday evening was entitled, "Historical Remarks on Some American Algebraists," and his Saturday morning address was on "Lie Algebras and Linear Groups."

The following papers were presented:

1. *A clan on the swastika*, by Professor Haskell Cohen, Louisiana State University.

2. *The sieve process for pairs of primes*, by Professor P. C. Garcia, University of Southwestern Louisiana.

For n an even positive integer, the natural number x is said to be the bottom half of a prime n pair if both x and $x+n$ are prime. This paper is concerned with finding all the bottom halves of prime n pairs less than or equal to a given natural number N by a process similar to the sieve method due to Erastosthenes. Included are counting processes that allow one to compute the number of bottom halves of prime n pairs $\leqq N$

3. *Associativity property of finite groups*, by Professor W. E. Koss, Louisiana Polytechnic Institute.

If a set of n elements satisfy the closure, identity, and inverse properties of a group with respect to a well defined operation, then at least $2n(n-1)(n-2)$ products of the form $a_ia_ja_k$ must be checked if the exhaustion of the cases method is used to check for associativity. The method devised in this note reduces the problem to one where only $(n-1)^2$ permutation products of the form P_iP_j are needed.

4. *Curves of positive area*, by Professor Gail Young, Tulane University.

5. *A special application of interior ballistics*, by Professor D. E. Johnson, Louisiana Polytechnic Institute.

6. *Bessel polynomials*, by Professor C. W. Barnes, University of Mississippi.

7. *Differentiability of solutions of* $Y'=f(X, Y)$ by Professor D. R. Scholz, Louisiana State University.

8. *On the convergence of the ratio of Fibonacci numbers*, by Professor W. M. Sanders, Mississippi Southern College.

Let $\{a_n\}$ be the Fibonacci sequence defined by the recursive formula $a_0=0$, $a_1=1, \cdots$, $a_n=a_{n-1}+a_{n-2}$ for $n\geq 2$. The infinite series $\sum_{n=1}1/a_n$, where a_n is the nth Fibonacci number, converges. This is established by the ratio test after having resolved the sequence of ratios $\{a_n/a_{n+1}\}$ into two bounded subsequences one of which is strictly monotone increasing and the other strictly monotone decreasing having the same limit. No explicit usage of the well-known fact that $\lim (a_n/a_{n+1})=2/(1+\sqrt{5})$ is made.

9. *On the automorphism group of a class of finite p-groups*, by Professor O. J. Huval, University of Southwestern Louisiana.

In this article it is shown that if a finite p-group G possesses an abelian subgroup M whose

factor group is cyclic, then G has outer automorphisms. If in addition the exponent of the center of G is less than or equal to the index of M in G, then the order of the automorphism group of G is divisible by the order of G.

10. *On subgroups which commute with every subgroup*, by Professor R. B. Howe, Mississippi State University.

It is shown that the condition that a subgroup H of a group G commute with every subgroup of G is a necessary but not a sufficient condition that H be normal in G.

11. *Rings which are unions of fields*, by Professor J. W. Ellis, Louisiana State University, New Orleans.

Z. L. LOFLIN, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The forty-first regular meeting of the Southern California Section of the Mathematical Association of America was held at the University of California, Santa Barbara, California, on March 11, 1961. Professor R. C. James, Chairman of the Section, presided. The registered attendance was 176, including 130 members of the Association.

At the business meeting Professor T. M. Apostol, Chairman of the Nominating Committee, reported that the following officers were elected by mail ballot for the next academic year: Chairman, Professor Clifford Bell, University of California, Los Angeles; Vice-Chairman, Professor P. A. White, University of Southern California; Secretary-Treasurer, Mr. R. B. Herrera, Los Angeles City College. Professor Apostol reported also that the members of the Program Committee elected for the coming year were: Professor Howard Tucker, University of California, Riverside, Chairman; Dr. Richard Bellman, Rand Corporation; Professor Richard Dean, Caltech; Professor Anthony Mardellis, Long Beach State College; Mr. T. E. Sydnor, Pasadena City College.

Mr. B. K. Gold, Chairman of the Contest Committee, presented a report for the committee and a motion, which was approved, requesting the National Mathematics Contest Office to invite all high schools in the Southern California area to participate in the 1962 National Contest.

Professor W. E. Smith of Occidental College presented a resolution honoring two former governors of the Section, who are retiring this year: Professor P. H. Daus, University of California, Los Angeles, and Professor C. G. Jaeger, Pomona College. Professor Smith's resolution was approved with applause.

The following program was presented:

1. *The Fibonacci operator*, by Professor C. J. A. Halberg, Jr., University of California, Riverside.

A specific bounded linear operator on the sequence l_1 is considered. Certain techniques are used to determine the spectrum of this operator, and it is shown that the uniform norm of the nth power of the operator is equal to the $(n+2)$th term of the Fibonacci sequence.

2. *Wife selection problems*, by Professor T. S. Ferguson, University of California, Los Angeles.

N distinct concealed numbers are being shown to you one at a time in random order. You may stop the proceedings at any time by selecting the number being shown to you. The problem of finding a selection procedure to maximize some criterion may be interpreted in terms of selecting a wife, buying a house, choosing a new staff member, and so on. The well-known problem of maximizing the probability of selecting the largest of the N numbers when you know nothing whatsoever about their values is the starting point for the investigation of several related problems.

3. *The School Mathematics Study Group*, by Mr. William Wooton, Los Angeles Pierce College.

The School Mathematics Study Group has finished one major part of its work, that of providing a complete set of class-room tested model textbooks for grades seven through twelve. Projects still in progress are the production of monographs for students, study guides for teachers, and

model textual material for noncollege-bound students. Other projects include textbooks for elementary grades four through six, and an alternate treatment of high school geometry.

4. *From my problem-solving seminar*, by Professor George Pólya, Stanford University.

One of the teacher's tasks—perhaps his principal task—is to recognize and develop the right attitude to independent work (to problem solving) in his students. Therefore, the teacher's training should provide suitable opportunity for independent work, that is, "research" on the appropriate level. Such training of teachers is usually not done, but the speaker's seminar attempts to provide this opportunity. Three problems from the seminar are presented; one will appear in *The Mathematics Teacher*.

5. *Inequalities*, by Professor Robert Schatten, University of Southern California.

6. *Rational approximation problems*, by Dr. E. W. Cheney, Space Technology Laboratories.

An efficient computational algorithm, developed jointly by H. L. Loeb and the speaker, is presented for the following problem: Given elements, $F, G_1, \cdots, G_n, H_1, \cdots, H_m$, of the Banach space $C[0, 1]$, obtain the coefficients a_i and b_i which render the "rational" function $\sum a_i G_i / \sum b_i H_i$ an optimum approximation to F.

R. B. HERRERA, *Secretary*

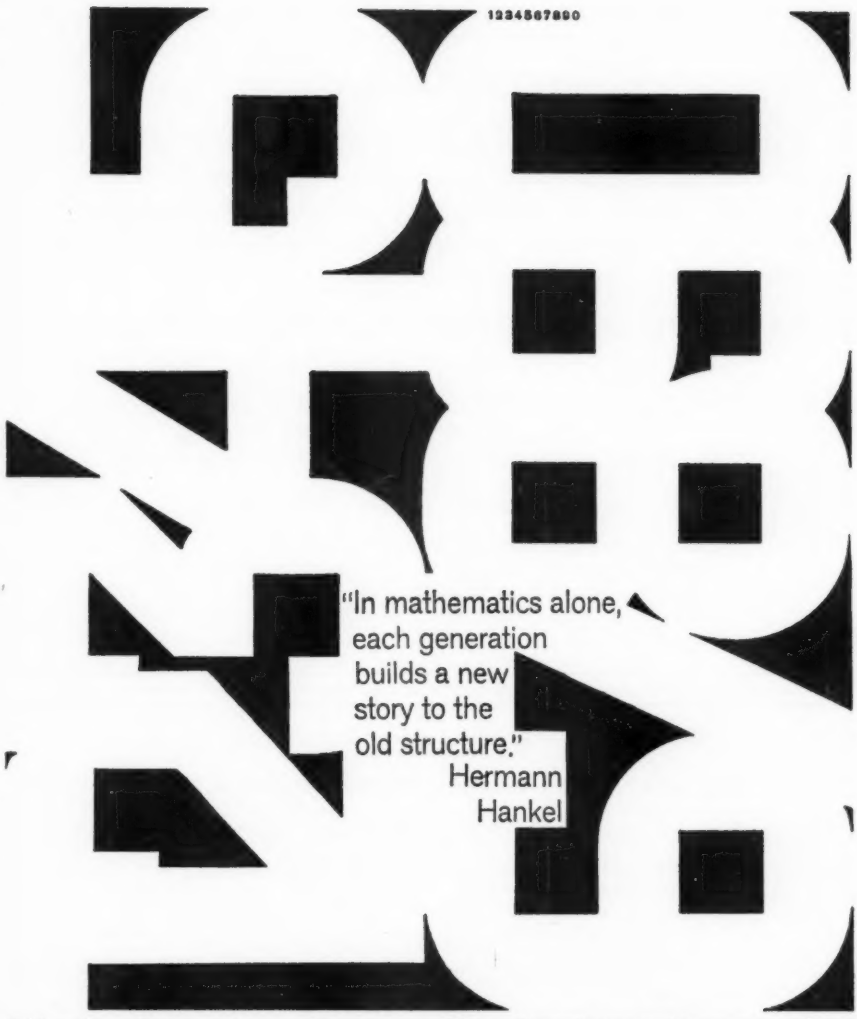
CALENDAR OF FUTURE MEETINGS

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–30, 1961.

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Tulane University, New Orleans, Louisiana, February 16–17, 1962.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN, University of Michigan, Ann Arbor, March 24, 1962.
MINNESOTA, Moorhead State College, Fall, 1961.
MISSOURI
NEBRASKA, University of Nebraska, Lincoln, April 13–14, 1962.
NEW JERSEY, St. Peter's College, Jersey City, November 4, 1961
NORTHEASTERN
NORTHERN CALIFORNIA, University of California, Davis, January 13, 1962.
OHIO
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Ursinus College, Collegeville, Pennsylvania, November 25, 1961.
ROCKY MOUNTAIN
SOUTHEASTERN, Woman's College, University of North Carolina, Greensboro, March 30–31, 1962.
SOUTHERN CALIFORNIA, Long Beach State College, March 9, 1962.
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN



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